Ringifying intersection cohomology

Tamás Hausel

Institute of Science and Technology Austria
http://hausel.ist.ac.at

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Der Wissenschaftsfonds.

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- Ringify! Find graded $H^*(X)$-algebra structure on $IH^*(X)$

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- *equivariantly formal*

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- for μ not weight multiplicity free we construct big (maximal) commutative subalgebra $Z(\mathcal{C}^\mu) \subset \mathcal{B}^\mu \subset \mathcal{C}^\mu$

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- $c_i \in \mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} \cong H^{2*}_{\mathrm{SL}_n}$ invariant polynomial

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Theorem (Hausel–Zveryk, Hausel 2022)

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Picture of $\operatorname{Spec}(\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2))$

- (Hausel–Rychlewicz 2022) computes $H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^n) \cong \mathcal{B}^{n\omega_1}(\mathfrak{sl}_2)$

$$\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2) \cong \begin{cases} \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2c_2)\ldots(M_1^2+4c_2)M_1\big) & \text{for } n \text{ even;} \\ \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2c_2)\ldots(M_1^2+9c_2)(M_1^2+c_2)\big) & \text{for } n \text{ odd.} \end{cases}$$

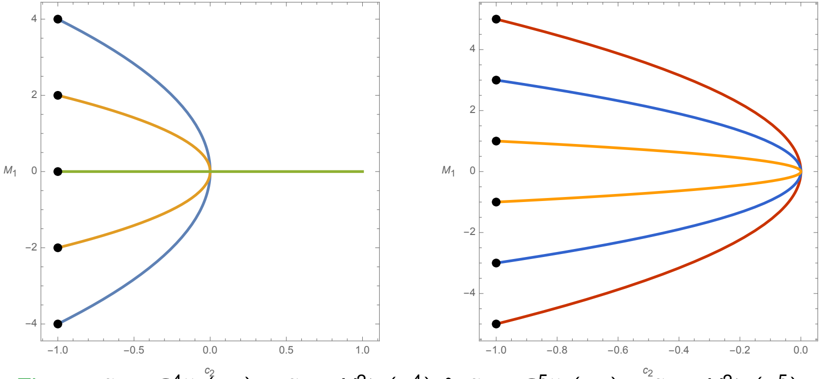

Figure: $\operatorname{Spec}\mathcal{B}^{4\omega_1}(\mathfrak{sl}_2) \cong \operatorname{Spec}H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^4)$ & $\operatorname{Spec}\mathcal{B}^{5\omega_1}(\mathfrak{sl}_2) \cong \operatorname{Spec}H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^5)$.

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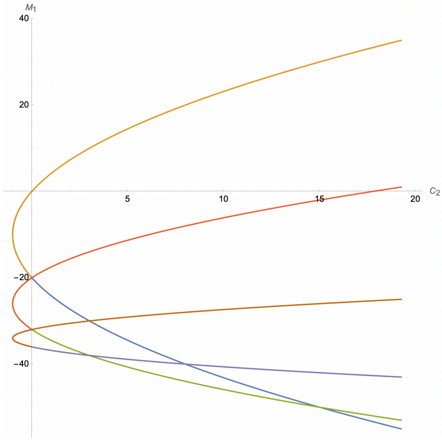

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

Endomorphisms of tensor product $M_\lambda \otimes V^\mu$

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$\mathcal{Z}^\mu = Z \otimes Z/$
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Examples of infinitesimal characters in tensor product

$$M_{-1} \otimes V^{5\omega_1} = P_{-6} \oplus P_{-4} \oplus P_{-2}$$

$$M_0 \otimes V^{5\omega_1} = P_{-5} \oplus P_{-3} \oplus M_{-1} \oplus M_5$$

$$M_1 \otimes V^{5\omega_1} = P_{-4} \oplus P_{-2} \oplus M_4 \oplus M_6$$

$$M_2 \otimes V^{5\omega_1} = \begin{array}{l} P_{-3} \oplus M_{-1} \oplus M_1 \oplus M_3 \\ \oplus M_5 \oplus M_7 \end{array}$$

$$M_3 \otimes V^{5\omega_1} = \begin{array}{l} P_{-2} \oplus M_0 \oplus M_2 \oplus M_4 \\ \oplus M_6 \oplus M_8 \end{array}$$

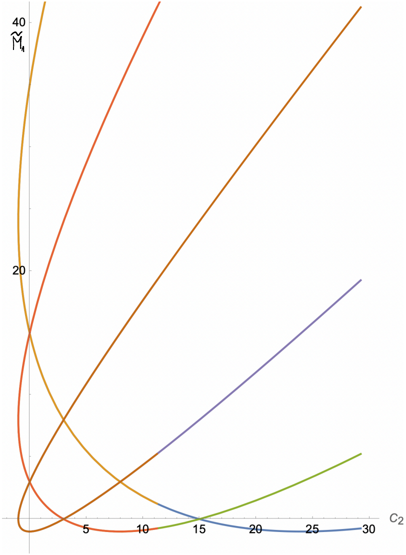

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

Happy birthday András!

First HAGS 2013, Chateau de Chillon