Anatomy of big algebras

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Big commmutative subalgebra of Kirillov algebra

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- for μ not weight multiplicity free we construct big (maximal) commutative subalgebra $Z(\mathcal{C}^\mu) \subset \mathcal{B}^\mu \subset \mathcal{C}^\mu$

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Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative

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Theorem (Hausel–Zveryk, Hausel 2022)

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Theorem (Hausel–Zveryk, Hausel 2022)

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Geometric properties of $\mathscr{B}^\mu$

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  e.g. $\mathrm{G} = \mathrm{SL}_2$, $X_\mu = \mathbb{P}(\mathbb{V}^\mu) \rightsquigarrow H^{2*}_{\mathrm{SL}_2}(\mathbb{P}(\mathbb{V}^\mu)) \cong \mathcal{C}^\mu \cong H^{2*}_{\mathrm{PGL}_2}(\mathrm{Gr}^\mu)$

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- $\mathrm{Gr}^\mu := \overline{\mathrm{PGL}_n[[z]]z^\mu} \subset \mathrm{Gr}$ affine Schubert; $\mathrm{Gr}^{\omega_k} = \mathrm{Gr}(k,n)$
- (Bezrukavnikov–Finkelberg 2008) describe the graded $H^*_{\mathrm{PGL}_n}$-algebra $H^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$ & $IH^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$ as module over it $\rightsquigarrow$

Corollary (Hausel 2022)

$H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu) \cong \mathcal{M}^\mu$ as $H^{2*}_{\mathrm{PGL}_n}$-algebras
$\mathrm{End}_{H^{2*}_{\mathrm{PGL}}(\mathrm{Gr}^\mu)}(IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \cong \mathcal{C}^\mu$
$IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu) \cong \mathcal{B}^\mu$ as $\mathcal{M}^\mu$-modules
$\rightsquigarrow$ graded $H^{2*}_{\mathrm{PGL}_n}$-algebra structure on $IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$

- for μ minuscule $H^{2*}_{\mathrm{G}}(\mathrm{G}/\mathrm{P}_\mu) \cong \mathcal{C}^\mu \cong H^{2*}_{\mathrm{G}^\vee}(\mathrm{Gr}^\mu) \cong H^{2*}_{\mathrm{G}^\vee}(\mathrm{G}^\vee/P^\vee_\mu)$
- for μ weight multiplicity free $H^{2*}_{\mathrm{G}}(X_\mu) \cong \mathcal{C}^\mu \cong H^{2*}_{\mathrm{G}^\vee}(\mathrm{Gr}^\mu)$
  e.g. $\mathrm{G} = \mathrm{SL}_2$, $X_\mu = \mathbb{P}(\mathbb{V}^\mu) \rightsquigarrow H^{2*}_{\mathrm{SL}_2}(\mathbb{P}(\mathbb{V}^\mu)) \cong \mathcal{C}^\mu \cong H^{2*}_{\mathrm{PGL}_2}(\mathrm{Gr}^\mu)$
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Visualizing big algebras

Visualizing big algebras

- $\mathrm{Spec}(\mathscr{B}^{\mu})$ affine scheme

Visualizing big algebras

- $\mathrm{Spec}(\mathcal{B}^\mu)$ affine scheme finite flat / $\mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{g}//\mathrm{G} \cong \mathfrak{t}//W$

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- $3 \rightsquigarrow$ (Ginzburg, 2008) $H^*(\mathrm{Gr}^\mu) \cong \mathcal{M}^\mu_{e_0}$ as assoc. graded of F

Picture of $\mathrm{Spec}(\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2))$

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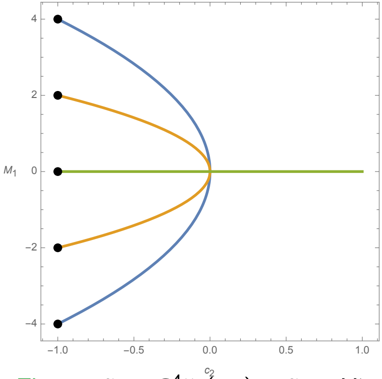
$$\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2) \cong \begin{cases} \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 4c_2)M_1\big) & \text{for } n \text{ even;} \\ \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 9c_2)(M_1^2 + c_2)\big) & \text{for } n \text{ odd.} \end{cases}$$

Picture of $\operatorname{Spec}(\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2))$

- (Hausel–Rychlewicz 2022) computes $H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^n) \cong \mathcal{B}^{n\omega_1}(\mathfrak{sl}_2)$

$$\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2) \cong \begin{cases} \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 4c_2)M_1\big) & \text{for } n \text{ even;} \\ \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 9c_2)(M_1^2 + c_2)\big) & \text{for } n \text{ odd.} \end{cases}$$



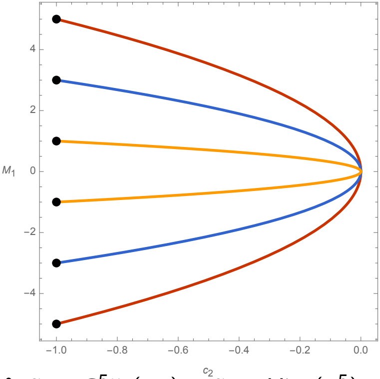


Figure: $\operatorname{Spec}\mathcal{B}^{4\omega_1}(\mathfrak{sl}_2) \cong \operatorname{Spec} H^*_{\mathrm{SL}_2}(\mathbb{P}^4)$ & $\operatorname{Spec}\mathcal{B}^{5\omega_1}(\mathfrak{sl}_2) \cong \operatorname{Spec} H^*_{\mathrm{SL}_2}(\mathbb{P}^5)$.

Picture of $\mathrm{Spec}(\mathcal{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

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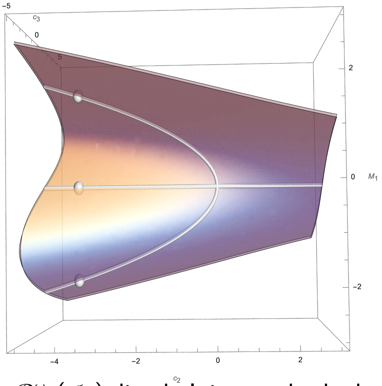

Figure: $\mathrm{Spec}\mathcal{B}^{\omega_1}(\mathfrak{sl}_3)$, its skeleton and principal spectrum

Picture of $\mathrm{Spec}(\mathcal{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

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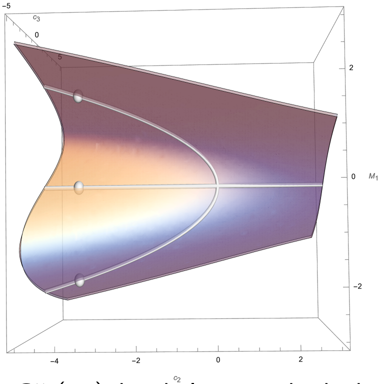

Figure: $\mathrm{Spec}\mathcal{B}^{\omega_1}(\mathfrak{sl}_3)$, its skeleton and principal spectrum

- principal spectrum can be identified with up,strange,down quarks in Gell-Mann's eightfold way

Skeleton of $\mathcal{S}\mathrm{pec}(\mathscr{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

- (Hausel–Rychlewic 2022) $\rightsquigarrow$
  $\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(X_{3\omega_1}) \cong H^{2*}_{\mathrm{SL}_3}((\mathbb{P}^2)^3/S_3)$

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  $\mathbb{C}[c_2, c_3, M_1, M_2]/\begin{pmatrix} M_1^4 - 6M_1^2M_2 + 4M_1^2c_2 - 18M_1c_3 + 3M_2^2 - 6M_2c_2, \\ M_1^3M_2 + M_1^3c_2 + 3M_1^2c_3 - 3M_1M_2^2 + M_1M_2c_2 + 4M_1c_2^2 - 9M_2c_3 \end{pmatrix}$

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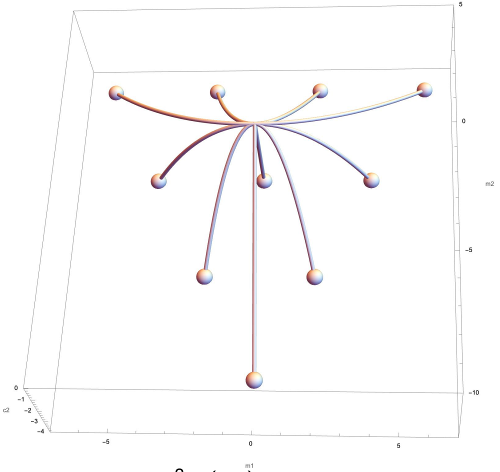

Figure: skeleton of $\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3)$ over its principal spectrum

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

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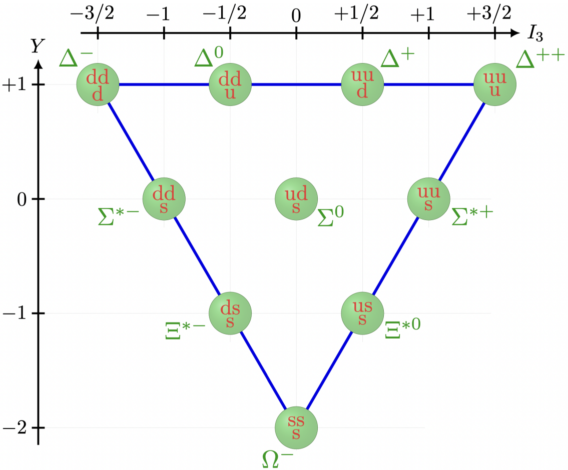

Figure: particles in baryon decuplet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

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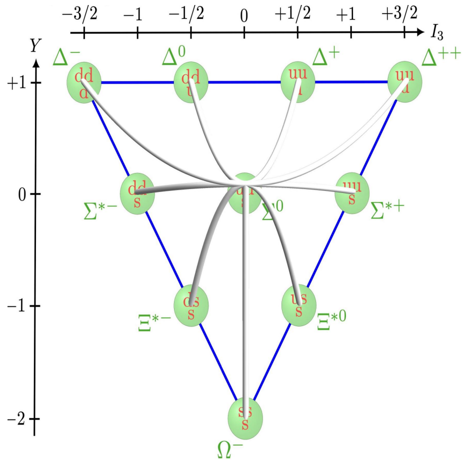

Figure: skeleton over baryon decuplet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

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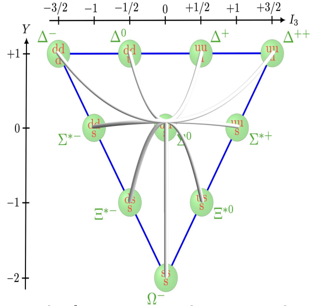

Figure: skeleton over baryon decuplet

- $\rightsquigarrow$ relations for $I_3 = \frac{(M_1)_{h_0}}{4}$ and $Y = \frac{(M_2)_{h_0}}{4}$ in baryon decuplet

$$I_3(Y-1)(4I_3^2 - 3Y - 4) = 0$$
$$16I_3^4 - 24I_3^2 Y - 16I_3^2 + 3Y^2 + 6Y = 0$$

Skeleton of $\mathcal{S}\mathrm{pec}(\mathscr{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

$$\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, N_1]/\begin{pmatrix} 3M_1^2 + N_1^2 + 12c_2, \\ M_1^3N_1 + c_2M_1N_1 - 9c_3M_1 \end{pmatrix}$$

$$\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, M_2]/\begin{pmatrix} M_1^2M_2 + c_2M_2 + 3c_3M_1, \\ M_1^4 + 4c_2M_1^2 + 3M_2^2, \\ 3M_1M_2^2 + 9c_3M_2 - c_2M_1^3 - 4c_2^2M_1 \end{pmatrix}$$

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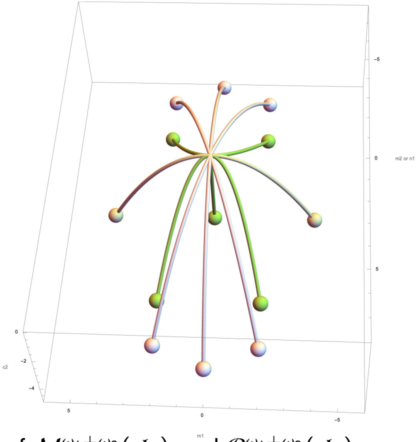

Figure: skeleton of $\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ and $\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ over principal spectra

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

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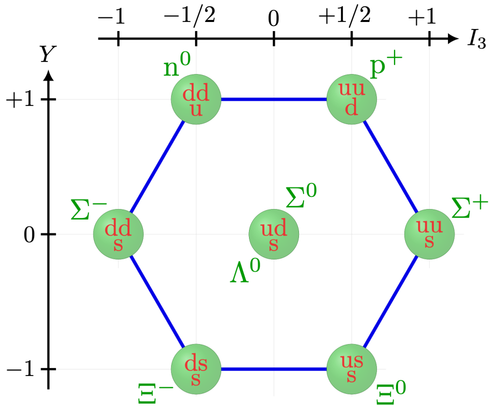

Figure: particles in baryon octet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

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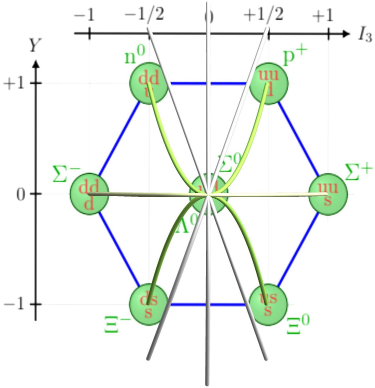

Figure: big and medium skeleton over baryon octet

Skeleton of $\mathcal{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2,\ M_2 := Dc_3/2,\ N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

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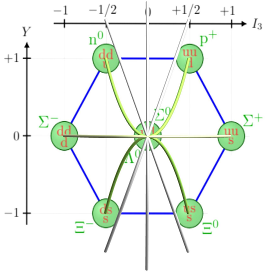

Figure: big and medium skeleton over baryon octet

- $\rightsquigarrow$ polynomial relations between I_3 and Y in baryon octet

$$Y(2I_3 - 1)(2I_3 + 1) = 0$$
$$4I_3^3 + 3I_3Y^2 - 4I_3 = 0$$
$$16I_3^4 - 16I_3^2 + 3Y^2 = 0$$

Nerves and crystals in big algebra

- $h_0 \in \mathfrak{t}_+ \subset \mathfrak{t}_\mathbb{R}$ in dominant Weyl chamber

Nerves and crystals in big algebra

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Nerves and crystals in big algebra

- $h_0 \in \mathfrak{t}_+ \subset \mathfrak{t}_\mathbb{R}$ in dominant Weyl chamber
- for $J \subset I := \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ face of dominant Weyl chamber: $\mathfrak{t}_+^J := \{\chi \in \mathfrak{t}_\mathbb{R} | \alpha_i(\chi) > 0$ if $i \in J$ and $\alpha_i(\chi) = 0$ o.w.$\}$

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Nerves and crystals in big algebra

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Nerves and crystals in big algebra

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  for $\pi_\mu : \mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(H_{\mathrm{G}}^{2*}) \cong \mathfrak{t}//W$

Nerves and crystals in big algebra

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  $h_J = \mathrm{pr}_{\mathfrak{t}_+^J}(h_0)$ orthogonal projection to $\mathfrak{t}_+^J$
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  for $\pi_\mu : \mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{t}/\!/W$
- monodromy along $n_{\{\alpha_i\}} \rightsquigarrow \mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C}) \to \mathrm{Spec}(\mathcal{B}^\mu_{h_{\{\alpha_i\}}})(\mathbb{C})$

Nerves and crystals in big algebra

- $h_0 \in \mathfrak{t}_+ \subset \mathfrak{t}_\mathbb{R}$ in dominant Weyl chamber
- for $J \subset I := \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ face of dominant Weyl chamber: $\mathfrak{t}_+^J := \{\chi \in \mathfrak{t}_\mathbb{R} | \alpha_i(\chi) > 0$ if $i \in J$ and $\alpha_i(\chi) = 0$ o.w.$\}$
  $h_J = \mathrm{pr}_{\mathfrak{t}_+^J}(h_0)$ orthogonal projection to $\mathfrak{t}_+^J$
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- *J-nerve* is the real curve $n_J := \pi_\mu^{-1}(l_j) \subset \mathrm{Spec}(\mathcal{B}^\mu)$ for $\pi_\mu : \mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{t}//W$
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Nerves and crystals in big algebra

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Nerves and crystals in big algebra

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Nerves and crystals in big algebra

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Corollary (Hausel–Rybnikov 2024)

The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$

Nerves and crystals in big algebra

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Corollary (Hausel–Rybnikov 2024)

The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$ induced by monodromy along the nerves $n_{\{\alpha_i\}}$

Nerves and crystals in big algebra

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  $+$ height function $(M_1)_{h_0} : \mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C}) \to \mathbb{Z} \rightsquigarrow$
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Corollary (Hausel–Rybnikov 2024)

The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$ induced by monodromy along the nerves $n_{\{\alpha_i\}}$ determines a structure of a μ-highest weight Kashiwara crystal

Nerves and crystals in big algebra

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Corollary (Hausel–Rybnikov 2024)

The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$ induced by monodromy along the nerves $n_{\{\alpha_i\}}$ determines a structure of a μ-highest weight Kashiwara crystal, $\mathrm{Spec}(\mathcal{B}^\mu_{h_0}) \to \mathrm{Spec}(\mathcal{M}^\mu_{h_0}) \subset X^*$ corresponding to the weight map.