

Anatomy of big algebras

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Der Wissenschaftsfonds.



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- for μ not weight multiplicity free we construct big (maximal) commutative subalgebra $Z(C^\mu) \subset \mathcal{B}^\mu \subset C^\mu$

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Theorem (Hausel–Zveryk, Hausel 2022)

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Theorem (Hausel–Zveryk, Hausel 2022)

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e.g. $G = \text{SL}_2$, $X_\mu = \mathbb{P}(\mathbb{V}^\mu) \leadsto H_{\text{SL}_2}^{2*}(\mathbb{P}(\mathbb{V}^\mu)) \cong C^\mu \cong H_{\text{PGL}_2}^{2*}(\text{Gr}^\mu)$
- for non-weight multiplicity free μ topological/geometrical description of ring structure on $IH_{\text{PGL}_n}^*(\text{Gr}^\mu)$ is open

Visualizing big algebras

Visualizing big algebras

- $\text{Spec}(\mathcal{B}^u)$ affine scheme

Visualizing big algebras

- $\mathrm{Spec}(\mathcal{B}^u)$ affine scheme finite flat / $\mathrm{Spec}(H_G^{2*}) \cong \mathfrak{g} // G \cong \mathfrak{t} // W$

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- $3 \rightsquigarrow$ (Ginzburg, 2008) $H^*(\text{Gr}^\mu) \cong \mathcal{M}_{e_0}^\mu$ as assoc. graded of F

Picture of $\text{Spec}(\mathcal{B}^{n\omega_1}(\mathfrak{s}\mathfrak{l}_2))$

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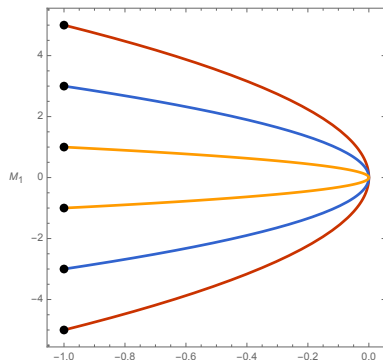
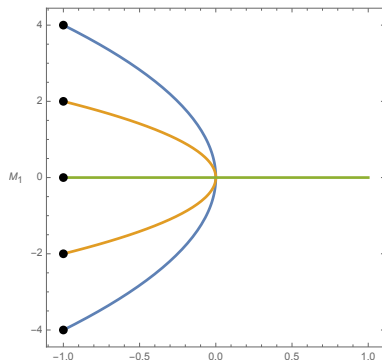


Figure: $\text{Spec} \mathcal{B}^{4\omega_1}(\mathfrak{sl}_2) \cong \text{Spec} H_{\text{SL}_2}^*(\mathbb{P}^4)$ & $\text{Spec} \mathcal{B}^{5\omega_1}(\mathfrak{sl}_2) \cong \text{Spec} H_{\text{SL}_2}^*(\mathbb{P}^5)$.

Picture of $\mathrm{Spec}(\mathcal{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

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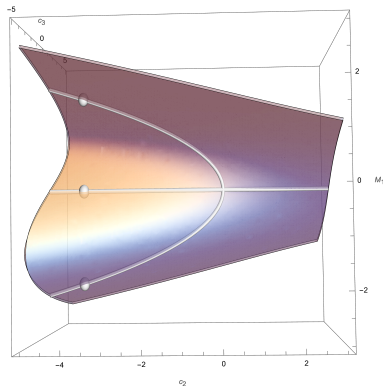


Figure: $\text{Spec}\mathcal{B}^{\omega_1}(\mathfrak{sl}_3)$, its skeleton and principal spectrum

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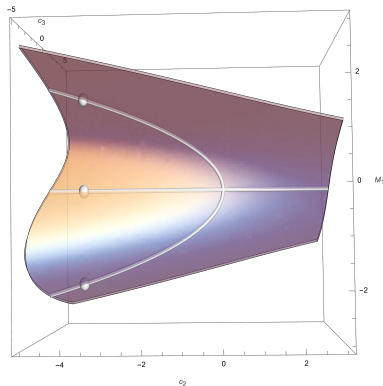


Figure: $\text{Spec} \mathcal{B}^{\omega_1}(\mathfrak{sl}_3)$, its skeleton and principal spectrum

- principal spectrum can be identified with up, strange, down quarks in Gell-Mann's eightfold way

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

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 $\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3) \cong H_{\mathrm{SL}_3}^{2*}(X_{3\omega_1}) \cong H_{\mathrm{SL}_3}^{2*}((\mathbb{P}^2)^3/S_3)$

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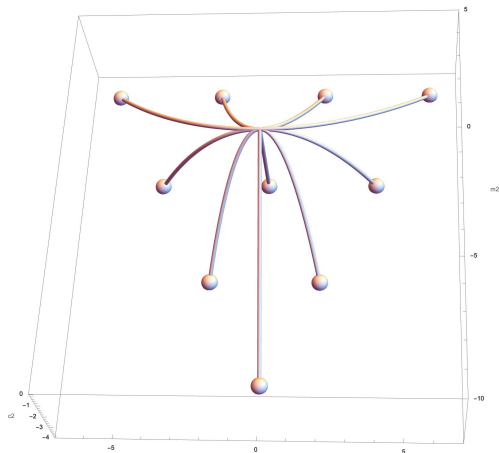


Figure: skeleton of $\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3)$ over its principal spectrum

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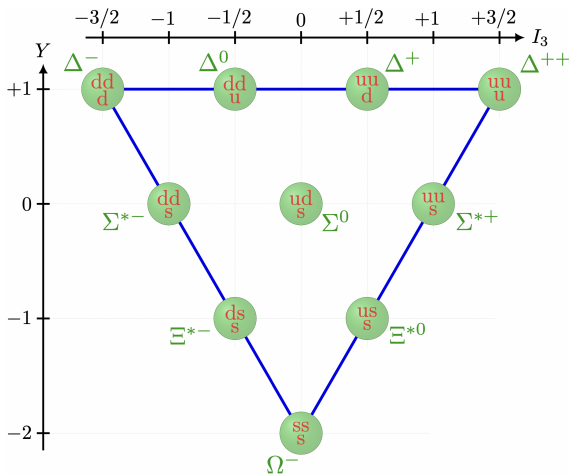


Figure: particles in baryon decuplet

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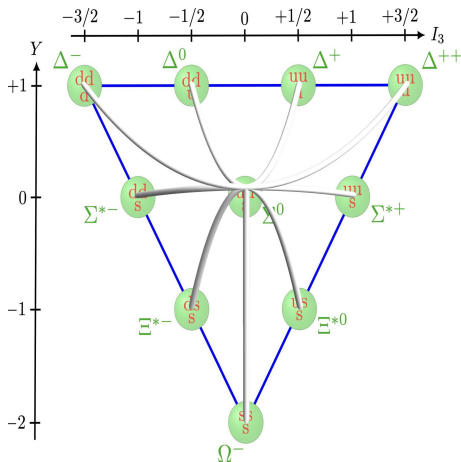


Figure: skeleton over baryon decuplet

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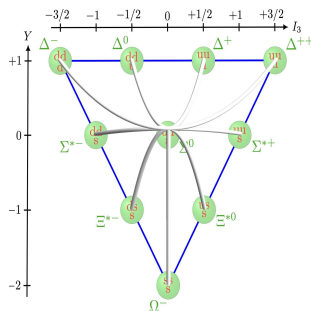


Figure: skeleton over baryon decuplet

- $\leadsto$ relations for $I_3 = \frac{(M_1)_{h_0}}{4}$ and $Y = \frac{(M_2)_{h_0}}{4}$ in baryon decuplet

$$\begin{aligned} I_3(Y-1)(4I_3^2 - 3Y - 4) &= 0 \\ 16I_3^4 - 24I_3^2 Y - 16I_3^2 + 3Y^2 + 6Y &= 0 \end{aligned}$$

Skeleton of $\text{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

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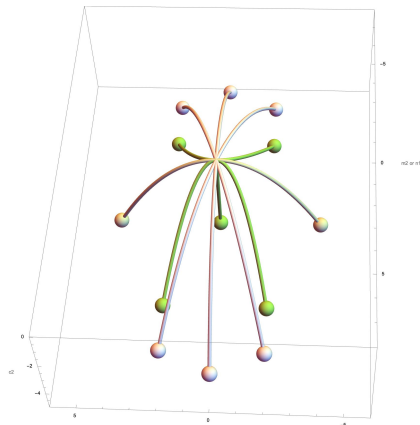


Figure: skeleton of $\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ and $\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ over principal spectra

Skeleton of $\text{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

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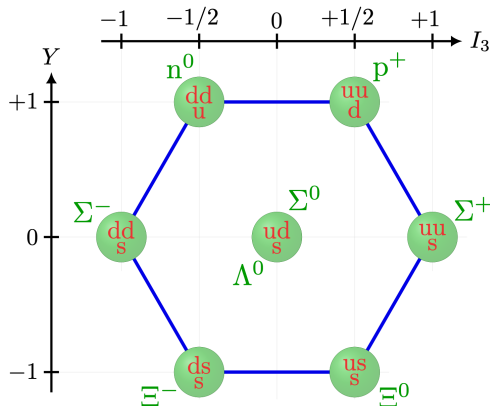


Figure: particles in baryon octet

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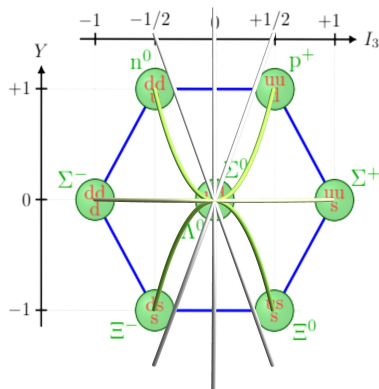


Figure: big and medium skeleton over baryon octet

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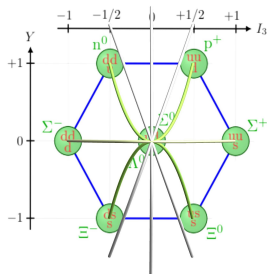


Figure: big and medium skeleton over baryon octet

- $\leadsto$ polynomial relations between I_3 and Y in baryon octet

$$Y(2I_3 - 1)(2I_3 + 1) = 0$$

$$4I_3^3 + 3I_3 Y^2 - 4I_3 = 0$$

$$16I_3^4 - 16I_3^2 + 3Y^2 = 0$$

Nerves and crystals in big algebra

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- (Halacheva–Kamnitzer–Rybnikov–Weekes 2020) $\leadsto$

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for $\pi_\mu : \text{Spec}(\mathcal{B}^\mu) \rightarrow \text{Spec}(H_G^{2*}) \cong \mathfrak{t} // W$
- monodromy along $n_{\{\alpha_i\}} \rightsquigarrow \text{Spec}(\mathcal{B}_{h_0}^\mu)(\mathbb{C}) \rightarrow \text{Spec}(\mathcal{B}_{h_{\{\alpha_i\}}}^\mu)(\mathbb{C})$
+ height function $(M_1)_{h_0} : \text{Spec}(\mathcal{B}_{h_0}^\mu)(\mathbb{C}) \rightarrow \mathbb{Z} \rightsquigarrow$
oriented graph on set $\text{Spec}(\mathcal{B}_{h_0}^\mu)(\mathbb{C})$
- (Halacheva–Kamnitzer–Rybnikov–Weekes 2020) $\rightsquigarrow$

Corollary (Hausel–Rybnikov 2024)

The I -colored oriented graph on $\text{Spec}(\mathcal{B}_{h_0}^\mu)(\mathbb{C})$

Nerves and crystals in big algebra

- $h_0 \in \mathfrak{t}_+ \subset \mathfrak{t}_{\mathbb{R}}$ in dominant Weyl chamber
- for $J \subset I := \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ face of dominant Weyl chamber:
 $\mathfrak{t}_+^J := \{\chi \in \mathfrak{t} \mid \alpha_i(\chi) > 0 \text{ if } i \in J \text{ and } \alpha_i(\chi) = 0 \text{ o.w.}\}$
 $h_J = \text{pr}_{\mathfrak{t}_+^J}(h_0)$ orthogonal projection to $\mathfrak{t}_+^J$
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Corollary (Hausel–Rybnikov 2024)

The I -colored oriented graph on $\text{Spec}(\mathcal{B}_{h_0}^\mu)(\mathbb{C})$ induced by monodromy along the nerves $n_{\{\alpha_i\}}$

Nerves and crystals in big algebra

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Corollary (Hausel–Rybnikov 2024)

The I -colored oriented graph on $\text{Spec}(\mathcal{B}_{h_0}^\mu)(\mathbb{C})$ induced by monodromy along the nerves $n_{\{\alpha_i\}}$ determines a structure of a μ -highest weight Kashiwara crystal

Nerves and crystals in big algebra

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Corollary (Hausel–Rybnikov 2024)

The I -colored oriented graph on $\text{Spec}(\mathcal{B}_{h_0}^\mu)(\mathbb{C})$ induced by monodromy along the nerves $n_{\{\alpha_i\}}$ determines a structure of a μ -highest weight Kashiwara crystal, $\text{Spec}(\mathcal{B}_{h_0}^\mu) \rightarrow \text{Spec}(\mathcal{M}_{h_0}^\mu) \subset X^$ corresponding to the weight map.*