Geometric representation theory from Higgs bundles

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Stratifications of Higgs bundle moduli spaces
and related topics
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Der Wissenschaftsfonds.

Mirror symmetry in the classical limit

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- G semisimple complex Lie group, concentrate on $\mathrm{G} = \mathrm{SL}_n$
- $X^*_+(\mathrm{G}) \ni \mu$-highest weight rep. $\rho^\mu : \mathrm{G} \to \mathrm{GL}(V^\mu)$
- $\mathcal{C}^\mu := (S(\mathfrak{g}^*) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}} \cong Maps(\mathfrak{g}, \mathrm{End}(V^\mu))^{\mathrm{G}}$
  associative, graded $S(\mathfrak{g}^*)^{\mathrm{G}} \cong H^{2*}_{\mathrm{G}}$-algebra: *Kirillov algebra*
  e.g. $M_1 := (X \mapsto Lie(\rho^\mu)(X))$ *small operator*
- (Kirillov 2000) $\mathcal{C}^\mu$ commutative $\Leftrightarrow$ ρ^μ weight multiplicity free;
  e.g. minuscule
- (Panyushev 2004) $\rightsquigarrow$ when μ minuscule, $\mathcal{C}^\mu \cong H^{2*}_{\mathrm{G}}(\mathrm{G}/P_\mu)$
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Theorem (Hausel 2022)

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{C}^{\omega_k}) \arrow[d] \arrow[dr, phantom, "\llcorner"] \arrow[rrr, bend left, "\cong"] & \mathrm{Spec}_{\mathbb{A}^\vee}(\Lambda^k(\mathbb{E}^\vee)_c) \arrow[l, twoheadrightarrow] \arrow[r, phantom, "\cong"] \arrow[d] & \mathcal{H}^{\omega_k}_c(W_0^+) \arrow[r, twoheadrightarrow] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n}))) \arrow[d] \\
\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrr, bend right, "\cong"'] & \mathbb{A}^\vee \arrow[l, twoheadrightarrow] \arrow[r, phantom, "\cong"] & \mathbb{A} \arrow[r, twoheadrightarrow] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\end{tikzcd}$$

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- for μ not weight multiplicity free we construct big (maximal) commutative subalgebra $Z(\mathcal{C}^\mu) \subset \mathcal{B}^\mu \subset \mathcal{C}^\mu$

Construction of big algebra

Construction of big algebra

- $\{X_i\}$ basis for SL_n

Construction of big algebra

- $\{X_i\}$ basis for SL_n, $\{X^i\}$ dual basis

Construction of big algebra

- $\{X_i\}$ basis for SL_n, $\{X^i\}$ dual basis wrt Killing form

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Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative

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Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative, cyclic

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Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ *is commutative, cyclic, finite-free* $/H^{2*}_{\mathrm{SL}_n}$

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Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative, cyclic, finite-free $/H^{2*}_{\mathrm{SL}_n}$ and Gorenstein

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Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative, cyclic, finite-free $/H^{2*}_{\mathrm{SL}_n}$ and Gorenstein

- proof via a universal big algebra $\mathcal{B} \subset (S(\mathfrak{g}) \otimes U(\mathfrak{g}))^{\mathrm{G}}$ as a Gaudin algebra from (Feigin–Frenkel, 1992)

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Theorem (Hausel–Zveryk, Hausel 2022)

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Conjecture (Hausel 2022)

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{B}^\mu) \arrow[d] \arrow[rrr, bend left, "\cong"] & \mathrm{Spec}_{\mathbb{A}^\vee}(\mathcal{B}^\mu(\rho^\mu(\mathbb{E}^\vee_c)))) \arrow[l, two heads] \arrow[d] \arrow[r, phantom, "\cong"] & \mathrm{Aff}(\mathcal{H}^\mu_c(\mathcal{O}_{W_0^+})) \arrow[r, two heads] \arrow[d] & \mathrm{Spec}(IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \arrow[d] \\
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\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrr, bend right, "\cong"'] & \mathbb{A}_{\mathrm{SL}_n} \arrow[l] \arrow[r, phantom, "\cong"] & \mathbb{A}_{\mathrm{PGL}_n} \arrow[r, two heads] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\end{tikzcd}$$

Picture of $\mathrm{Spec}(\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2))$

- (Hausel–Rychlewicz 2022) computes $H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^n) \cong \mathcal{B}^{n\omega_1}(\mathfrak{sl}_2)$

$$\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2) \cong \begin{cases} \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2+4c_2)M_1\big) & \text{for } n \text{ even;} \\ \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2+9c_2)(M_1^2+c_2)\big) & \text{for } n \text{ odd.} \end{cases}$$

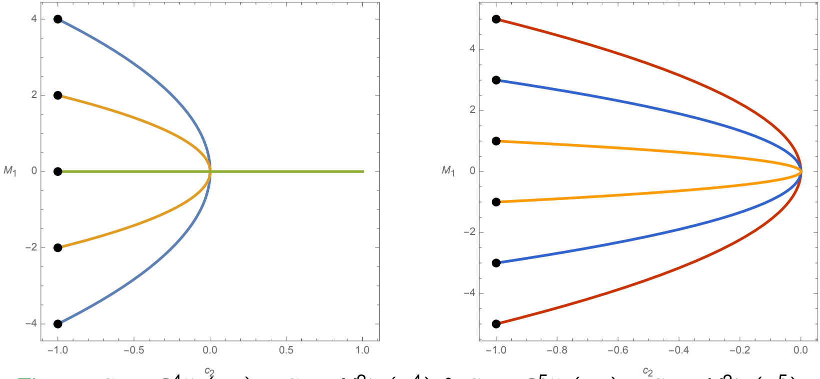

Figure: $\mathrm{Spec}\mathcal{B}^{4\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec} H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^4)$ & $\mathrm{Spec}\mathcal{B}^{5\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec} H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^5)$.

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Examples

- (Rozhkovskaya 2003) $\rightsquigarrow$
  $\mathcal{R}^\mu$ commutative$\Leftrightarrow$ $\mathcal{Z}^\mu = \mathcal{G}^\mu = \mathcal{R}^\mu \Leftrightarrow \rho_\mu$ w.m.f., e.g. minuscule
- ρ_μ minuscule $\Leftrightarrow$ loop rotation $\mathbb{C}^*$-action on Gr^μ is trivial $\rightsquigarrow$

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  $\mathcal{R}^{n\omega_1} \cong$
  $\mathbb{C}[C_2, M_1]/(\prod_{i=1}^n (M_1 - 4(j^2 - n/2 - jn + (n/2 - j)\sqrt{C_2 + 1})))$

(Rozhkovskaya 2003)'s example of $\mathfrak{sl}_2$ Kostant algebra

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$$\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2) \cong \mathbb{C}[C_2, M_1]/$$
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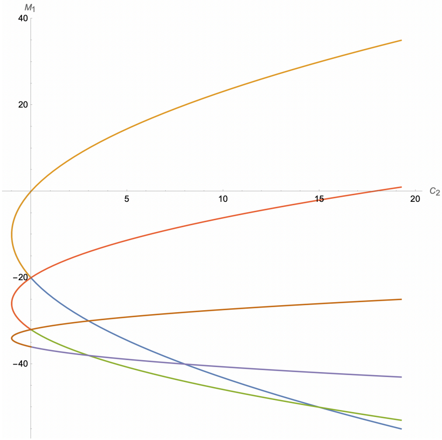

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

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Examples of infinitesimal characters in tensor product

$$M_{-1} \otimes V^{5\omega_1} = P_{-6} \oplus P_{-4} \oplus P_{-2}$$

$$M_0 \otimes V^{5\omega_1} = P_{-5} \oplus P_{-3} \oplus M_{-1} \oplus M_5$$

$$M_1 \otimes V^{5\omega_1} = P_{-4} \oplus P_{-2} \oplus M_4 \oplus M_6$$

$$M_2 \otimes V^{5\omega_1} = \begin{array}{l} P_{-3} \oplus M_{-1} \oplus M_1 \oplus M_3 \\ \oplus M_5 \oplus M_7 \end{array}$$

$$M_3 \otimes V^{5\omega_1} = \begin{array}{l} P_{-2} \oplus M_0 \oplus M_2 \oplus M_4 \\ \oplus M_6 \oplus M_8 \end{array}$$

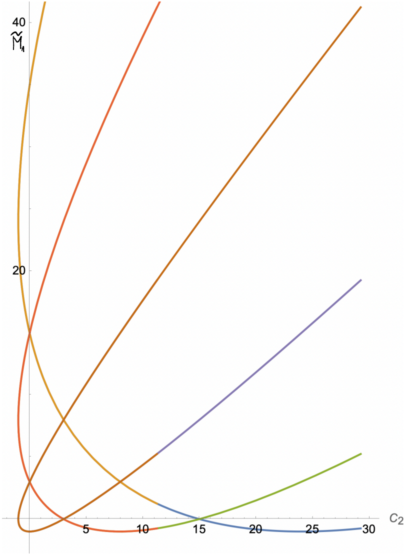

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

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- (preliminary work by (Gonzalez 2025))