Mirror symmetry and big algebras

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Der Wissenschaftsfonds.

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dual special Lagrangian fibrations

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- $\rightsquigarrow D^b_{coh}(\mathbb{M}_{\mathrm{DR}}(\mathrm{G})) \overset{GLC}{\cong} D^b_{D-mod}(Bun_{\mathrm{G}^\vee})$

  Geometric Langlands Corr. of (Beilinson–Drinfeld, 1995)

- (Donagi, Pantev 2012) explain classical limit of HMS:
  $\mathcal{S} : D^b_{coh}(\mathbb{M}_{\mathrm{G}}) \overset{CLHMS}{\cong} D^b_{coh}(\mathbb{M}_{\mathrm{G}^\vee})$
  generically as Fourier–Mukai transform
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- test for $\mathcal{O}_{\mathbb{M}_{\mathrm{G}^\vee}} \in D^b_{coh}(\mathbb{M}_{\mathrm{G}^\vee})$:
  $\mathcal{H}^{\mu}_{c}(\mathcal{S}(\mathcal{O}_{\mathbb{M}_{\mathrm{G}^\vee}})) = \mathcal{H}^{\mu}_{c}(\mathcal{O}_{W_0^+}) = \mathcal{S}(\mathcal{W}^{\mu}_{c}(\mathcal{O}_{\mathbb{M}_{\mathrm{G}^\vee}})) = \mathcal{S}(\rho^\mu(\mathbb{E}^\vee)_c)$
- the Hecke transform of the Hitchin section $\mathcal{H}^{\mu}_{c}(\mathcal{O}_{W_0^+})$ is supported at a union of Lagrangian upward flows

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- Motivating Problem: find coordinates s.t.

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Multiplicity algebra of $h_{\mathcal{E}}$

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- *multiplicity algebra*

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  - $\sum_i \dim(Q_{h_{\mathcal{E}}}^i) t^i = \frac{\chi_{\mathbb{C}^\times}(Sym(T_{\mathcal{E}}^{+*}))}{\chi_{\mathbb{C}^\times}(Sym(\mathbb{A}^*))} = m_{\mathcal{E}}(t) \in \mathbb{N}[t]$ monic, palindromic

Multiplicity algebra of $h_{\mathcal{E}}$

- *multiplicity algebra* of $h_{\mathcal{E}} = (h_1, .., h_N) : \mathbb{C}^N \cong W^+_{\mathcal{E}} \to \mathbb{A} \cong \mathbb{C}^N$:

  $Q_{h_{\mathcal{E}}} := \mathbb{C}[W^+_{\mathcal{E}} \cap h^{-1}(0)] = \mathbb{C}[W^+_{\mathcal{E}}]/(h^{-1}(\mathfrak{m}_0)) = \mathbb{C}[x_1, \ldots, x_N]/(h_1, \ldots, h_N)$
- notion due to (Arnold et al. 1982)
- $\dim(Q_{h_{\mathcal{E}}}) < \infty \Leftrightarrow h_{\mathcal{E}}$ is proper $\Leftrightarrow W^+_{\mathcal{E}} \subset \mathbb{M}_n$ closed $\Leftrightarrow W^+_{\mathcal{E}} \cap h^{-1}(0) = \{\mathcal{E}\} \Leftrightarrow \mathcal{E}$ *very stable*
- in this case $m_{\mathcal{E}} := \dim(Q_{h_{\mathcal{E}}})$ is the *multiplicity* of $h^{-1}_{\mathcal{E}}(0)$
- as $h_{\mathcal{E}}$ is $\mathbb{C}^\times$-equivariant $\rightsquigarrow \mathbb{C}^\times \circlearrowright Q_{h_{\mathcal{E}}} \rightsquigarrow Q_{h_{\mathcal{E}}} = \bigoplus_{k=0}^{m} Q^k_{h_{\mathcal{E}}}$ s.t.
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Hitchin map as spectrum of equivariant cohomology

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Equivariant cohomology from zero scheme

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Theorem (Hausel–Rychlewicz, 2022)

$\mathrm{G} \circlearrowright X$ *regular* $\rightsquigarrow$ *trace map* $\mathrm{tr} : H^*_{\mathrm{G}}(X; \mathbb{C}) \xrightarrow{\cong} \mathbb{C}[\mathcal{Z}]^{\mathrm{G}}$

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$\mathrm{G} \circlearrowright X$ *regular* $\rightsquigarrow$ *trace map* $\mathrm{tr}: H^*_{\mathrm{G}}(X;\mathbb{C}) \xrightarrow{\cong} \mathbb{C}[\mathcal{Z}]^{\mathrm{G}}$ *over* $H^{2*}_{\mathrm{G}} \cong \mathbb{C}[\mathfrak{g}]^{\mathrm{G}}$

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- construction by applying Kirillov M-operators to Φ_a

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- $k = 1$ familiar bundle of algebra structure from BNR corr.

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  e.g. $M_1 = \mathrm{Lie}(\rho^{\omega_k})$ small operator
- $(\Lambda^{n-k}(tI + A))^* = t^{n-k} + t^{n-k-1}B_1 + \dots B_{n-k} : \Lambda^k(\mathbb{C}^n) \to \Lambda^k(\mathbb{C}^n)$
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  $\det(tI + A)Id_{\Lambda^k\mathbb{C}^n} = (t^n + t^{n-2}c_2 + \cdots + c_n)Id_{\Lambda^k\mathbb{C}^n}$
- $\rightsquigarrow \mathcal{C}^{\omega_k} \cong \langle M_1, \dots, M_k, N_1, \dots, N_{n-k}\rangle_{H^*_{\mathrm{SL}_n}} \cong H^*_{\mathrm{SL}_n}(\mathrm{Gr}(k,n), \mathbb{C})$
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- *spectral curve* $C_a := \mathrm{Spec}_C(\mathcal{C}^{\omega_1}(\mathcal{E}_a)) \subset \mathrm{Spec}(\mathrm{Sym}^*(K))$

Minuscule Kirillov algebra

- $G = SL_n$, $\omega_k := \Lambda^k \mathbb{C}^n \in X_*^+(SL_n)$
- $A \in \mathfrak{sl}_n \rightsquigarrow \Lambda^k(tI + A) = t^k + t^{k-1}A_1 + \dots A_k : \Lambda^k(\mathbb{C}^n) \to \Lambda^k(\mathbb{C}^n)$
  define $M_i := (A \mapsto A_i) \in (Map(\mathfrak{g} \to \mathrm{End}(\Lambda^k(\mathbb{C}^n)))^G \cong \mathcal{C}^{\omega_k}$
  e.g. $M_1 = \mathrm{Lie}(\rho^{\omega_k})$ small operator
- $(\Lambda^{n-k}(tI + A))^* = t^{n-k} + t^{n-k-1}B_1 + \dots B_{n-k} : \Lambda^k(\mathbb{C}^n) \to \Lambda^k(\mathbb{C}^n)$
  define $N_j := (A \mapsto B_j) \in (Map(\mathfrak{g} \to \mathrm{End}(\Lambda^k(\mathbb{C}^n)))^G \cong \mathcal{C}^{\omega_k}$
- M_i, N_j are commuting operators satisfying:
  $(t^k + t^{k-1}A_1 + \cdots + A_k)(t^{n-k} + t^{n-k-1}B_1 + \cdots + B_{n-k}) =$
  $\det(tI + A)Id_{\Lambda^k\mathbb{C}^n} = (t^n + t^{n-2}c_2 + \cdots + c_n)Id_{\Lambda^k\mathbb{C}^n}$
- $\rightsquigarrow \mathcal{C}^{\omega_k} \cong \langle M_1, \dots, M_k, N_1, \dots, N_{n-k}\rangle_{H^*_{SL_n}} \cong H^*_{SL_n}(\mathrm{Gr}(k,n), \mathbb{C})$
- $E_0 = \mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{1-n}$ and $\mathcal{E}_a = (E_0, \Phi_a)$ for $a \in \mathbb{A}$
- $M_i(\Phi_a) \in H^0(\mathrm{End}(\Lambda^k E_0) \otimes K^i) \cong \mathrm{Hom}(K^{-i}, \mathrm{End}(\Lambda^k E_0)) \rightsquigarrow$
  $\tilde{\Lambda}^k(\mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{1-n}) \oplus \tilde{\Lambda}^k(E_0) = \tilde{\Lambda}^k(E_0) \oplus \tilde{\Lambda}^k(E_0) \to \tilde{\Lambda}^k(E_0)$
  $\rightsquigarrow$ bundle of algebra structure $\mathcal{C}^{\omega_k}(\mathcal{E}_a)$ on $\tilde{\Lambda}^k(E_0) = K^{\binom{k}{2}}\Lambda^k(E_0)$
- e.g. $k = 1 \rightsquigarrow M_1^i(\Phi_a) = \Phi_a^i : K^{-i} \to \mathrm{End}(E_0) \rightsquigarrow$ multiplication
  $(\mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{1-n}) \oplus E_0 = E_0 \oplus E_0 \to E_0 \rightsquigarrow \mathcal{C}^{\omega_1}(\mathcal{E}_a)$ on E_0
- *spectral curve* $C_a := \mathrm{Spec}_C(\mathcal{C}^{\omega_1}(\mathcal{E}_a)) \subset \mathrm{Spec}(\mathrm{Sym}^*(K)) \cong T^*C$

Minuscule attractor as universal spectral curve

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BNR correspondence from mirror symmetry

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  $S(\mathcal{O}_{\mathcal{E}})$ should be by FM transform a line bundle on $h^{-1}_{\mathrm{PGL}_n}(a)$

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- $\rightsquigarrow$ *classical limit of geometric Satake*

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Construction of the big algebra

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Theorem (Hausel–Zveryk, Hausel 2022)

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- $$\begin{array}{rcl} D : \mathcal{C}^\mu & \to & \mathcal{C}^\mu = (S(\mathfrak{sl}_n^*) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}} \\ A & \mapsto & \sum_i \rho^\mu(X^i) \frac{\partial(A)}{\partial X_i} \end{array}$$ Kirillov-Wei D-operator
- $c_i \in \mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} \cong H^*_{\mathrm{SL}_n}$ invariant polynomial
  $t^n + c_2(a)t^{n-2} + \cdots + c_n(a)$ char poly of $a \in \mathfrak{sl}_n$
- $M_{i-1} := D(c_i)$ Kirillov's M-operators *medium operators*
- $\mathcal{M}^\mu := \langle 1_{V^\mu}, M_1, \ldots, M_{n-1} \rangle_{H^*_{\mathrm{SL}_n}} \subset \mathcal{C}^\mu$ *medium algebra* $\cong Z(\mathcal{C}^\mu)$
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Conjecture (Hausel 2022)

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{B}^\mu) \arrow[d] \arrow[rrr, bend left, "\cong"] & \mathrm{Spec}_{\mathbb{A}^\vee}(\rho^\mu(\mathbb{E}^\vee_c)) \arrow[l, two heads] \arrow[dd] \arrow[r, phantom, "\cong"] & \mathcal{H}^\mu_c(W_0^+) \arrow[r, two heads] \arrow[d] & \mathrm{Spec}(IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \arrow[d] \\
\mathrm{Spec}(\mathcal{M}^\mu) \arrow[d] \arrow[r, phantom, "\llcorner"] & & \overline{W^+_\mu} \arrow[d] \arrow[r, phantom, "\lrcorner"] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \arrow[d] \\
\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrr, bend right, "\cong"'] & \mathbb{A}^\vee \arrow[l, two heads] \arrow[r, phantom, "\cong"] & \mathbb{A} \arrow[r, two heads] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\end{tikzcd}$$

Visualizing big algebras

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  $\mathcal{B}^\mu_{\mathrm{SL}_2} := \mathcal{B}^\mu \otimes_{H^{2*}_{\mathrm{G}}} H^{2*}_{\mathrm{SL}_2}$ and $\mathcal{M}^\mu_{\mathrm{SL}_2} := \mathcal{M}^\mu \otimes_{H^{2*}_{\mathrm{G}}} H^{2*}_{\mathrm{SL}_2}$
  principal big and medium algebras
- *big skeleton*: $\mathrm{Spec}(\mathcal{B}^\mu_{\mathrm{SL}_2}) \to \mathrm{Spec}(H^{2*}_{\mathrm{SL}_2}) \cong \mathbb{A}^1$
  medium skeleton: $\mathrm{Spec}(\mathcal{M}^\mu_{\mathrm{SL}_2}) \to \mathrm{Spec}(H^{2*}_{\mathrm{SL}_2}) \cong \mathbb{A}^1$
  big, medium principal spectrum: $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})$, $\mathrm{Spec}(\mathcal{M}^\mu_{h_0})$
  where $h_0 \in \mathfrak{sl}_2 = \langle e_0, h_0, f_0 \rangle \subset \mathfrak{g}$ is principal semisimple

Theorem (Hausel 2023)

1. $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})$ *reduced of length* $\dim(V^\mu)$
2. $\mathrm{Spec}(\mathcal{M}^\mu_{h_0}) = \mathrm{Spec}(\rho^\mu(U(\mathfrak{g}_{h_0})) = \{\lambda \in X^* : V^\mu_\lambda \neq 0\} \subset \mathfrak{t}^*$
3. $\mathcal{M}^\mu_{\mathrm{SL}_2} = Rees(F) = \oplus_{i=0}^\infty c_2^i F_i R$ *where F filtration on* $R = \mathcal{M}^\mu_{h_0}$
4. *multiplicity algebra* $Q^\mu_\lambda = \lim_{\epsilon\to 0} \mathrm{Spec}(\mathcal{B}^\mu_{\epsilon h_0})_\lambda \cong IH^{2*}(\mathcal{W}^\mu_\lambda)$

Visualizing big algebras

- $\mathrm{Spec}(\mathcal{B}^\mu)$ affine scheme finite flat / $\mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{g}//\mathrm{G} \cong \mathfrak{t}//W$
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Visualizing big algebras

- $\mathrm{Spec}(\mathcal{B}^\mu)$ affine scheme finite flat / $\mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{g}//\mathrm{G} \cong \mathfrak{t}//W$
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- 3 $\rightsquigarrow$ (Ginzburg, 2008) $H^*(\mathrm{Gr}^\mu) \cong \mathcal{M}^\mu_{e_0}$ as assoc. graded of F

Picture of $\mathrm{Spec}(\mathscr{B}^{n\omega_1}(\mathfrak{sl}_2))$

Picture of $\mathrm{Spec}(\mathscr{B}^{n\omega_1}(\mathfrak{sl}_2))$

- (Hausel–Rychlewicz 2022) computes $H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^n)$

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- (Hausel–Rychlewicz 2022) computes
  $H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^n) \cong \mathcal{B}^{n\omega_1}(\mathfrak{sl}_2) \cong H^{2*}_{\mathrm{SL}_2}(\mathrm{Gr}^{n\omega_1})$
  $$\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2) \cong \begin{cases} \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 4c_2)M_1\big) & \text{for } n \text{ even;} \\ \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 9c_2)(M_1^2 + c_2)\big) & \text{for } n \text{ odd.} \end{cases}$$

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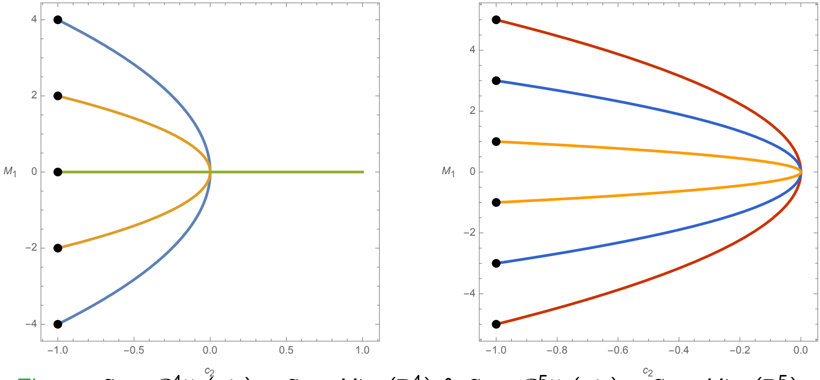

Figure: $\mathrm{Spec}\,\mathcal{B}^{4\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec}\,H^*_{\mathrm{SL}_2}(\mathbb{P}^4)$ & $\mathrm{Spec}\,\mathcal{B}^{5\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec}\,H^*_{\mathrm{SL}_2}(\mathbb{P}^5)$.

Picture of $\mathrm{Spec}(\mathscr{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

Picture of $\mathrm{Spec}(\mathcal{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

- (Panyushev 2004) $\rightsquigarrow \mathcal{B}^{\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(\mathbb{P}^2)$

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- (Panyushev 2004) $\rightsquigarrow \mathcal{B}^{\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(\mathbb{P}^2)$ or Cayley-Hamilton

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Picture of $\mathrm{Spec}(\mathcal{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

- (Panyushev 2004) $\rightsquigarrow \mathcal{B}^{\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(\mathbb{P}^2)$ or Cayley-Hamilton $\rightsquigarrow \mathcal{B}^{\omega_1}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1]/(M_1^3 + c_2 M_1 + c_3)$.
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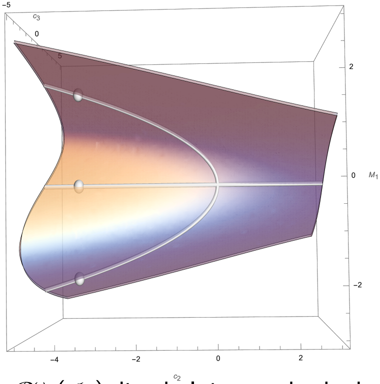

Figure: $\mathrm{Spec}\mathcal{B}^{\omega_1}(\mathfrak{sl}_3)$, its skeleton and principal spectrum

Picture of $\mathrm{Spec}(\mathcal{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

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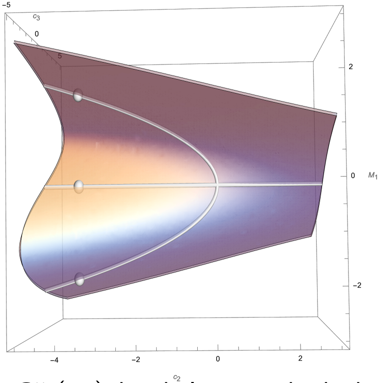

Figure: $\mathrm{Spec}\mathcal{B}^{\omega_1}(\mathfrak{sl}_3)$, its skeleton and principal spectrum

- principal spectrum can be identified with up,strange,down quarks in Gell-Mann's eightfold way

Skeleton of $\mathcal{S}\mathrm{pec}(\mathscr{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

- (Hausel–Rychlewic 2022) $\rightsquigarrow$
  $\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(X_{3\omega_1}) \cong H^{2*}_{\mathrm{SL}_3}((\mathbb{P}^2)^3/S_3)$

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

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  $\mathbb{C}[c_2, c_3, M_1, M_2]/\begin{pmatrix} M_1^4 - 6M_1^2M_2 + 4M_1^2c_2 - 18M_1c_3 + 3M_2^2 - 6M_2c_2, \\ M_1^3M_2 + M_1^3c_2 + 3M_1^2c_3 - 3M_1M_2^2 + M_1M_2c_2 + 4M_1c_2^2 - 9M_2c_3 \end{pmatrix}$

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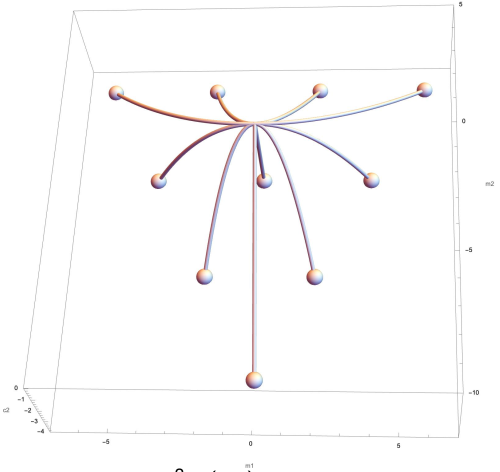

Figure: skeleton of $\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3)$ over its principal spectrum

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

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$\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(X_{3\omega_1}) \cong H^{2*}_{\mathrm{SL}_3}((\mathbb{P}^2)^3/S_3) \cong$
$\mathbb{C}[c_2, c_3, M_1, M_2]/\left(\begin{array}{l} M_1^4 - 6M_1^2M_2 + 4M_1^2c_2 - 18M_1c_3 + 3M_2^2 - 6M_2c_2, \\ M_1^3M_2 + M_1^3c_2 + 3M_1^2c_3 - 3M_1M_2^2 + M_1M_2c_2 + 4M_1c_2^2 - 9M_2c_3 \end{array}\right)$

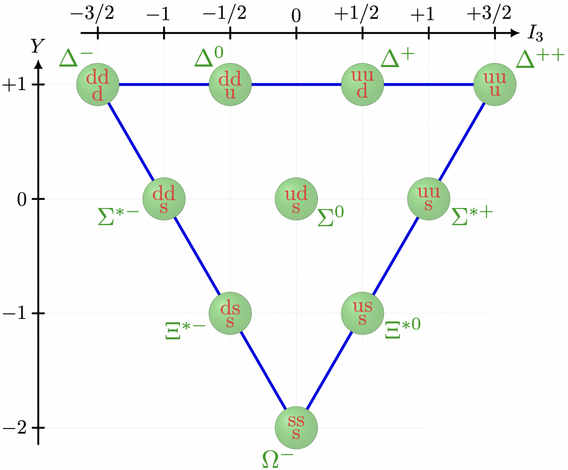

Figure: particles in baryon decuplet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

- (Hausel–Rychlewic 2022) $\rightsquigarrow$
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  $\mathbb{C}[c_2, c_3, M_1, M_2]/\left(\begin{array}{c} M_1^4 - 6M_1^2M_2 + 4M_1^2c_2 - 18M_1c_3 + 3M_2^2 - 6M_2c_2, \\ M_1^3M_2 + M_1^3c_2 + 3M_1^2c_3 - 3M_1M_2^2 + M_1M_2c_2 + 4M_1c_2^2 - 9M_2c_3 \end{array} \right)$

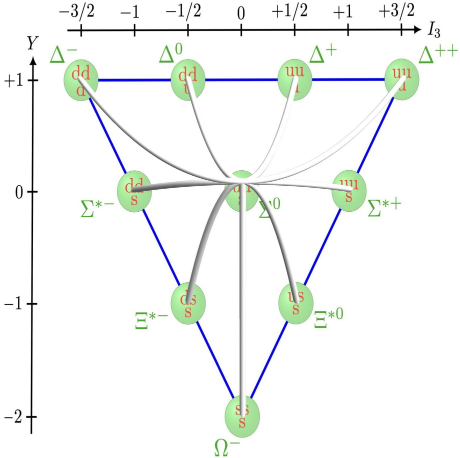

Figure: skeleton over baryon decuplet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

- (Hausel–Rychlewic 2022) $\rightsquigarrow$
  $\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(X_{3\omega_1}) \cong H^{2*}_{\mathrm{SL}_3}((\mathbb{P}^2)^3/S_3) \cong$
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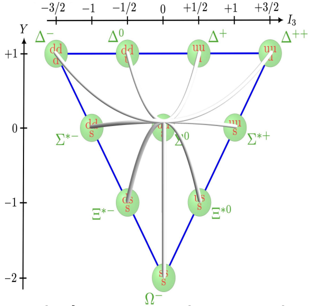

Figure: skeleton over baryon decuplet

- $\rightsquigarrow$ relations for $I_3 = \frac{(M_1)_{h_0}}{4}$ and $Y = \frac{(M_2)_{h_0}}{4}$ in baryon decuplet

$$I_3(Y-1)(4I_3^2 - 3Y - 4) = 0$$
$$16I_3^4 - 24I_3^2Y - 16I_3^2 + 3Y^2 + 6Y = 0$$

Skeleton of $\mathcal{S}\mathrm{pec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

$$\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, N_1]/\begin{pmatrix} 3M_1^2 + N_1^2 + 12c_2, \\ M_1^3 N_1 + c_2 M_1 N_1 - 9c_3 M_1 \end{pmatrix}$$

$$\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, M_2]/\begin{pmatrix} M_1^2 M_2 + c_2 M_2 + 3c_3 M_1, \\ M_1^4 + 4c_2 M_1^2 + 3M_2^2, \\ 3M_1 M_2^2 + 9c_3 M_2 - c_2 M_1^3 - 4c_2^2 M_1 \end{pmatrix}$$

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

$$\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, N_1]/\left(\begin{array}{l} 3M_1^2 + N_1^2 + 12c_2, \\ M_1^3 N_1 + c_2 M_1 N_1 - 9c_3 M_1 \end{array}\right)$$

$$\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, M_2]/\left(\begin{array}{l} M_1^2 M_2 + c_2 M_2 + 3c_3 M_1, \\ M_1^4 + 4c_2 M_1^2 + 3M_2^2, \\ 3M_1 M_2^2 + 9c_3 M_2 - c_2 M_1^3 - 4c_2^2 M_1 \end{array}\right)$$

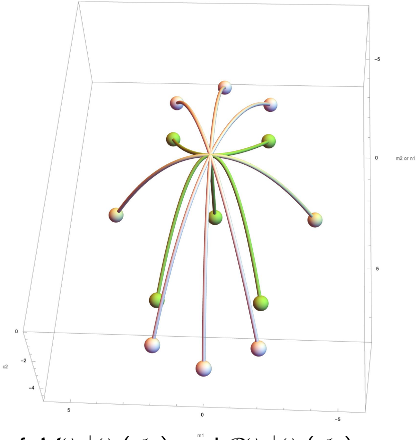

Figure: skeleton of $\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ and $\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ over principal spectra

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

$$\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, N_1]/\begin{pmatrix} 3M_1^2 + N_1^2 + 12c_2, \\ M_1^3N_1 + c_2M_1N_1 - 9c_3M_1 \end{pmatrix}$$

$$\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, M_2]/\begin{pmatrix} M_1^2M_2 + c_2M_2 + 3c_3M_1, \\ M_1^4 + 4c_2M_1^2 + 3M_2^2, \\ 3M_1M_2^2 + 9c_3M_2 - c_2M_1^3 - 4c_2^2M_1 \end{pmatrix}$$

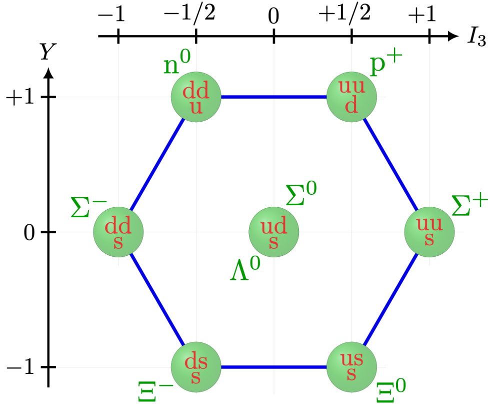

Figure: particles in baryon octet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

$$\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, N_1]/\begin{pmatrix} 3M_1^2 + N_1^2 + 12c_2, \\ M_1^3N_1 + c_2M_1N_1 - 9c_3M_1 \end{pmatrix}$$

$$\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, M_2]/\begin{pmatrix} M_1^2M_2 + c_2M_2 + 3c_3M_1, \\ M_1^4 + 4c_2M_1^2 + 3M_2^2, \\ 3M_1M_2^2 + 9c_3M_2 - c_2M_1^3 - 4c_2^2M_1 \end{pmatrix}$$

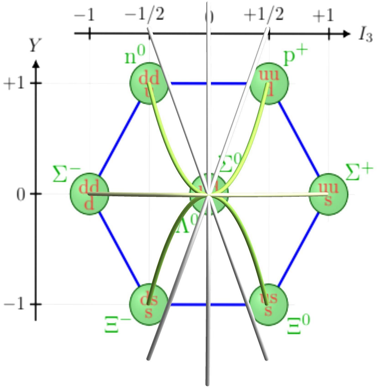

Figure: big and medium skeleton over baryon octet

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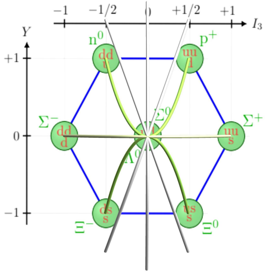

Figure: big and medium skeleton over baryon octet

- $\rightsquigarrow$ polynomial relations between I_3 and Y in baryon octet

$$Y(2I_3 - 1)(2I_3 + 1) = 0$$
$$4I_3^3 + 3I_3Y^2 - 4I_3 = 0$$
$$16I_3^4 - 16I_3^2 + 3Y^2 = 0$$

Nerves and crystals in big algebra

Nerves and crystals in big algebra

- $h_0 \in \mathfrak{t}_+ \subset \mathfrak{t}_{\mathbb{R}}$ in dominant Weyl chamber

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Nerves and crystals in big algebra

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  $\mathfrak{h}^J_+ := \{\chi \in \mathfrak{h}_\mathbb{R} | \alpha_i(\chi) > 0$ if $i \in J$ and $\alpha_i(\chi) = 0$ o.w.$\}$
  $h_J = \mathrm{pr}_{\mathfrak{h}^J_+}(h_0)$ orthogonal projection to $\mathfrak{h}^J_+$
  $l_J = \overline{h_0 h_J} \subset \mathfrak{h}_\mathbb{R}$ real segment connecting h_0 and h_J
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Corollary (Hausel–Rybnikov 2024)

The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$

Nerves and crystals in big algebra

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Corollary (Hausel–Rybnikov 2024)

The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$ *induced by monodromy along the nerves* $n_{\{\alpha_i\}}$

Nerves and crystals in big algebra

- $h_0 \in \mathfrak{t}_+ \subset \mathfrak{t}_{\mathbb{R}}$ in dominant Weyl chamber
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  $h_J = \mathrm{pr}_{\mathfrak{h}_+^J}(h_0)$ orthogonal projection to $\mathfrak{h}_+^J$
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- *J-nerve* is the real curve $n_J := \pi_\mu^{-1}(I_j) \subset \mathrm{Spec}(\mathcal{B}^\mu)$
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Corollary (Hausel–Rybnikov 2024)

The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$ induced by monodromy along the nerves $n_{\{\alpha_i\}}$ determines a structure of a μ-highest weight Kashiwara crystal

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The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$ induced by monodromy along the nerves $n_{\{\alpha_i\}}$ determines a structure of a μ-highest weight Kashiwara crystal, $\mathrm{Spec}(\mathcal{B}^\mu_{h_0}) \to \mathrm{Spec}(\mathcal{M}^\mu_{h_0}) \subset X^*$ corresponding to the weight map.