Ringifying intersection cohomology

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  e.g. minuscule
- G acts on $X \rightsquigarrow$ *equivariant intersection cohomology* $IH_{\mathrm{G}}^*(X)$ a module over *equivariant cohomology ring* $H_{\mathrm{G}}^*(X)$ over
  $H_{\mathrm{G}}^*(pt) = H_{\mathrm{G}}^* = \mathbb{C}[\mathfrak{g}]^{\mathrm{G}} = \mathbb{C}[\mathfrak{h}]^W$ polynomial ring
- *equivariantly formal* $\leftrightarrow H_{\mathrm{G}}^*(X)$ and $IH_{\mathrm{G}}^*(X)$ are free over
  $H_{\mathrm{G}}^*(X) \leftrightarrow IH^*(X) \cong IH_{\mathrm{G}}^*(X) \otimes_{H_{\mathrm{G}}^*} \mathbb{C}$
- (Panyushev 2004) $\rightsquigarrow$ when μ minuscule, $\mathcal{C}^\mu \cong H_{\mathrm{G}}^{2*}(\mathrm{G}/P_\mu)$
  $\rightsquigarrow$ finite free over H_{G}^{2*} and $\mathcal{C}_x^\mu \subset \mathrm{End}(V^\mu)$ cyclic for $x \in \mathfrak{g}^{reg}$
- for μ not weight multiplicity free we construct big (maximal) commutative subalgebra $Z(\mathcal{C}^\mu) \subset \mathcal{B}^\mu \subset \mathcal{C}^\mu$

Construction of big algebra

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- $\{X_i\}$ basis for SL_n

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Theorem (Hausel–Zveryk, Hausel 2022, Ngô 2025)

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Theorem (Hausel–Zveryk, Hausel 2022, Ngô 2025)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative, cyclic

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Theorem (Hausel–Zveryk, Hausel 2022, Ngô 2025)

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- proof via a universal big algebra $\mathcal{B} \subset (S(\mathfrak{g}) \otimes U(\mathfrak{g}))^{\mathrm{G}}$ as a Gaudin algebra from (Feigin–Frenkel, 1992)

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Construction of big algebra

- $\{X_i\}$ basis for SL_n, $\{X^i\}$ dual basis wrt Killing form
- $$\begin{array}{rccc} D: \mathcal{C}^\mu & \to & \mathcal{C}^\mu = (S(\mathfrak{sl}_n^*) \otimes \mathrm{End}(V^\mu))^{\mathrm{SL}_n} \\ A & \mapsto & \sum_i \rho^\mu(X^i) \frac{\partial(A)}{\partial X_i} \end{array}$$ Kirillov-Wei operator
- $c_i \in \mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} \cong H^{2*}_{\mathrm{SL}_n}$ invariant polynomial
  $t^n + c_2(a)t^{n-2} + \cdots + c_n(a)$ char poly of $a \in \mathfrak{sl}_n$
- $M_{i-1} := D(c_i)$ Kirillov's M-operators *medium operators*
- $\mathcal{M}^\mu := \langle 1_{V^\mu}, M_1, \ldots, M_{n-1}\rangle_{H^{2*}_{\mathrm{SL}_n}} \subset \mathcal{C}^\mu$ *medium algebra* $\subseteq Z(\mathcal{C}^\mu)$
- $B_{k,i-k} := D^k(c_i) \in \mathcal{C}^\mu$ *big operators* of age k and degree $i-k$
- $\mathcal{B}^\mu := \langle 1_{V^\mu}, \{B_{k,i-k}\}_{k,i}\rangle_{H^{2*}_{\mathrm{SL}_n}} \subset \mathcal{C}^\mu$ *big algebra* $\supset \mathcal{M}^\mu$

Theorem (Hausel–Zveryk, Hausel 2022, Ngô 2025)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative, cyclic, finite-free $/H^{2*}_{\mathrm{SL}_n}$ and Gorenstein

- proof via a universal big algebra $\mathcal{B} \subset (S(\mathfrak{g}) \otimes U(\mathfrak{g}))^{\mathrm{G}}$ as a Gaudin algebra from (Feigin–Frenkel, 1992)
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  $\rightsquigarrow \mathcal{B}^\mu$ is a quantum integrable system

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  satisfies corollary $\rightsquigarrow$
  $H^*_{\mathrm{SO}_5}(\mathrm{Gr}^{2\omega_2})$-algebra structure on $IH^*_{\mathrm{SO}_5}(\mathrm{Gr}^{2\omega_2})$ is not unique

Picture of $\mathrm{Spec}(\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2))$

- (Hausel–Rychlewicz 2022) computes $H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^n) \cong \mathcal{B}^{n\omega_1}(\mathfrak{sl}_2)$

$$\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2) \cong \begin{cases} \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 4c_2)M_1\big) & \text{for } n \text{ even;} \\ \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 9c_2)(M_1^2 + c_2)\big) & \text{for } n \text{ odd.} \end{cases}$$

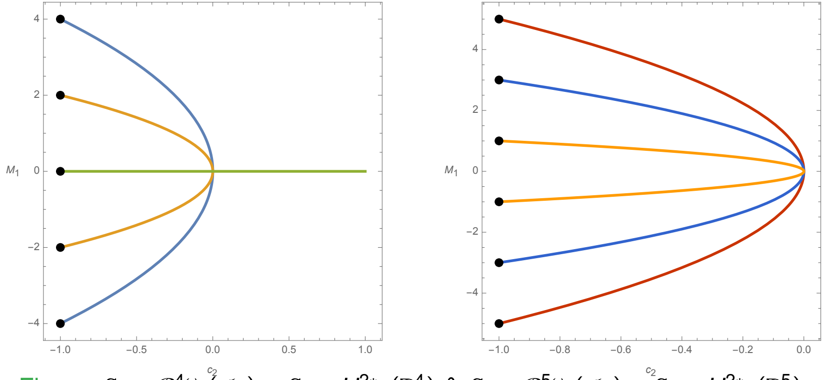

Figure: $\mathrm{Spec}\mathcal{B}^{4\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec} H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^4)$ & $\mathrm{Spec}\mathcal{B}^{5\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec} H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^5)$.

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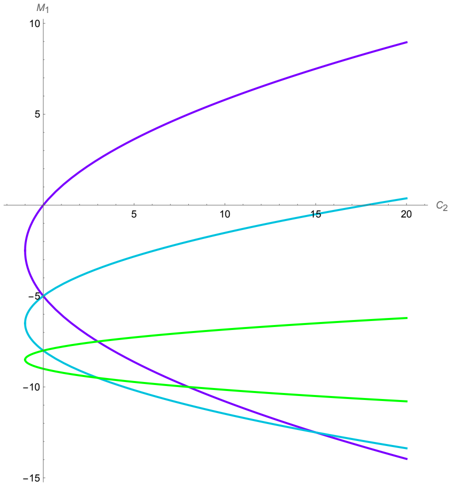

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

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Corollary (Hausel 2025)

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Examples of infinitesimal characters in tensor product

$$M_{-1} \otimes V^{5\omega_1} = P_{-6} \oplus P_{-4} \oplus P_{-2}$$

$$M_0 \otimes V^{5\omega_1} = P_{-5} \oplus P_{-3} \oplus M_{-1} \oplus M_5$$

$$M_1 \otimes V^{5\omega_1} = P_{-4} \oplus P_{-2} \oplus M_4 \oplus M_6$$

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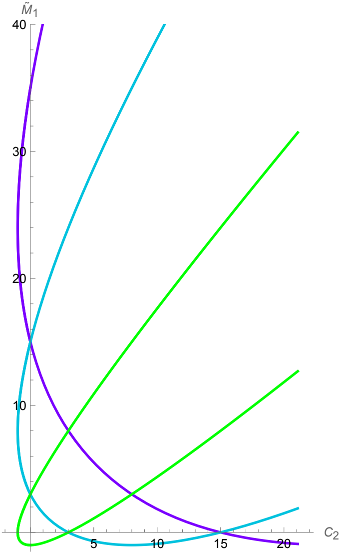

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

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