Higgs bundle tournaments

based on joint project with Mirko Mauri

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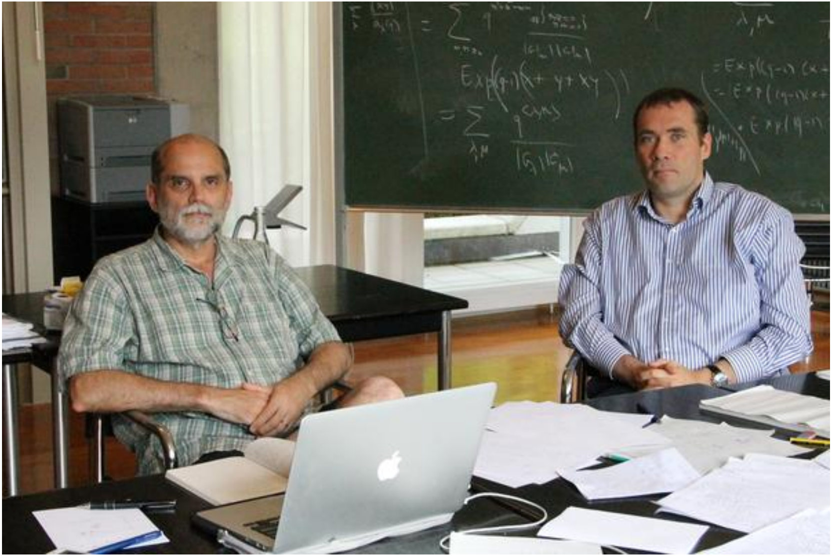
Der Wissenschaftsfonds.

Oberwolfach RIP 2008 and 2014



Oberwolfach RIP 2008 and 2014

AN AMERICAN TOURNAMENT TREATED BY THE CALCULUS OF SYMMETRIC FUNCTIONS.

By MAJOR P. A. MACMAHON.

PART I.

1. IN a tournament of n players, where each player plays every other player, there are $\frac{1}{2}n(n-1)$ games. Since each game may be won or lost there are $2^{\frac{1}{2}n(n-1)}$ events and I propose to analyse them by means of the powerful calculus of symmetric functions. The final result of the play is that the players are arranged in a definite order, each with a certain number of games to his credit. These numbers constitute a partition of the number $\frac{1}{2}n(n-1)$, and we may ask how many of the $2^{\frac{1}{2}n(n-1)}$ events will yield a given partition of $\frac{1}{2}n(n-1)$ when the players are or are not in an assigned order.

2. Consider the symmetric function

$$(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\ldots(\alpha_{n-1}+\alpha_n)$$

of the n quantities $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

It involves $\frac{1}{2}n(n-1)$ factors, and the terms, after carrying out the multiplication, are grouped together in monomial symmetric functions.

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- $(\deg(E), n) = 1 \rightsquigarrow$ similar combinatorics by (Villegas 2011, 2023, Reineke 2012, Rayan 2018)

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  $R := \{\pm(e_i - e_j)\}_{1\le i<j\le n} = R_+ \coprod R_- \subset \Lambda$ type A_{n-1} root system

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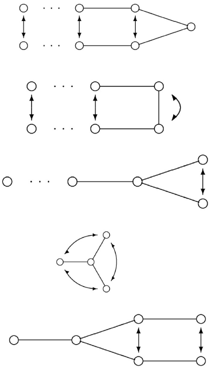

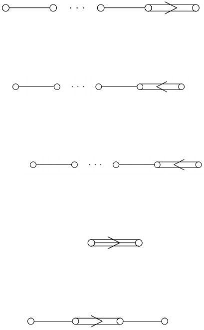

R	σ	R^σ	$\mathfrak{g}^\vee_{R^\sigma} \subset \mathfrak{g}^\vee_R$
A_{2n-1}		B_n	$\mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n}$
A_{2n}		C_n	$\mathfrak{so}_{2n+1} \subset \mathfrak{sl}_{2n+1}$
D_n		C_{n-1}	$\mathfrak{so}_{2n-1} \subset \mathfrak{so}_{2n}$
D_4		G_2	$\mathfrak{g}_2 \subset \mathfrak{so}_8$
E_6		F_4	$\mathfrak{f}_4 \subset \mathfrak{e}_6$

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  $\mathcal{B}^\mu \subset (S(\mathfrak{g}) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}}$ *big algebra*
  commutative, graded, cyclic, $H^{2*}_{\mathrm{G}} = \mathbb{C}[\mathfrak{g}]^{\mathrm{G}}$-algebra
  $\sigma : \mathrm{G} \to \mathrm{G}$ Dynkin automorphism, $\mu \in \Lambda^+(\mathrm{G}_\sigma) \rightsquigarrow$
  folding: $\mathcal{B}^\mu(\mathfrak{g})_\sigma \cong \mathcal{B}^\mu(\mathfrak{g}_\sigma)$ (Hausel, Zveryk 2023)
- $\mathrm{Spec}(\mathcal{B}^\mu \otimes_{H^{2*}_{\mathrm{SL}_n}} H^{2*}_{\mathrm{SL}_2}) \to \mathrm{Spec}(H^{2*}_{\mathrm{SL}_2}) \cong \mathbb{A}^1$ *big skeleton*
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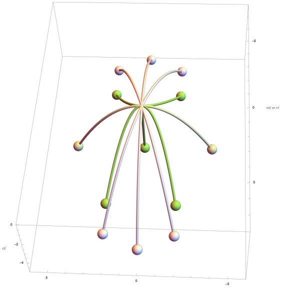

Figure: skeleton of $\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ and $\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ over principal spectra

Skeletons of big algebras

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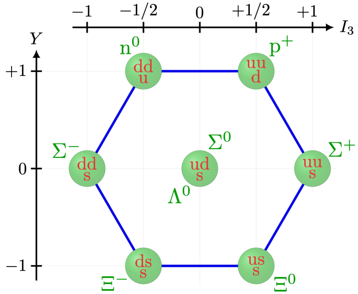

Figure: particles in baryon octet

Skeletons of big algebras

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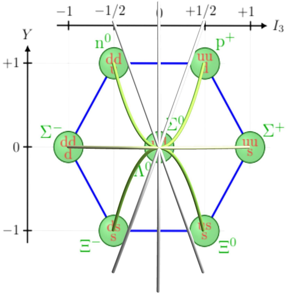

Figure: big and medium skeleton over baryon octet


ARITHMETIQUE GEOMETRIE
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DESSIN DE L'ARCHITECTURE
ANCIENNE
ECOLE ROYALE
DE
DESSIN
1767
DESSIN DES FLEURS ORNEMENTS
ANIMAUX ET FIGURES HUMAINES
SCULPTURE D'ORNEMENT
7

Arithmétique Géométrie - happy mathematics Fernando!