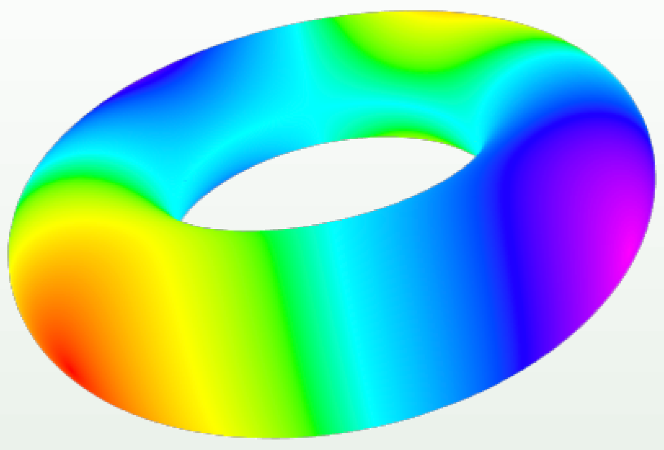


Hausel Group: Geometry and Its Interfaces

IST AUSTRIA 10 YEARS

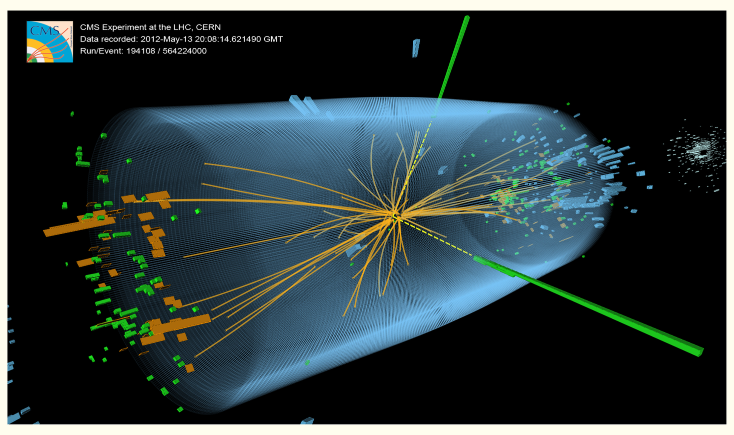
Institute of Science and Technology

Motivation

How can we understand spaces too large for traditional analysis? Combining ideas from representation theory and combinatorics, the Hausel group develops tools to study the topology of spaces arising from string theory and quantum field theory.

Suppose you have many particles, and consider the space made up of all the ways each particle can move between two points. Now, play the same game with more complicated objects, such as vector fields. The resulting spaces are too large to analyze, but it is possible to simplify them along structural symmetries, giving rise to moduli spaces that are finite-dimensional, but non-compact — again, defying traditional methods. The Hausel group studies the topology, geometry, and arithmetic of these moduli spaces, which include the moduli spaces of Yang-Mills instantons in four dimensions, and Higgs bundles in two dimensions, among others.

A *Higgs bundle*, just as its distant cousin the Higgs boson, is a connection together with an auxiliary Higgs field. Their moduli space on a Riemann surface has an intricate geometry with deep connections to integrable systems, number theory, and knot invariants. The following is one of our central problems.

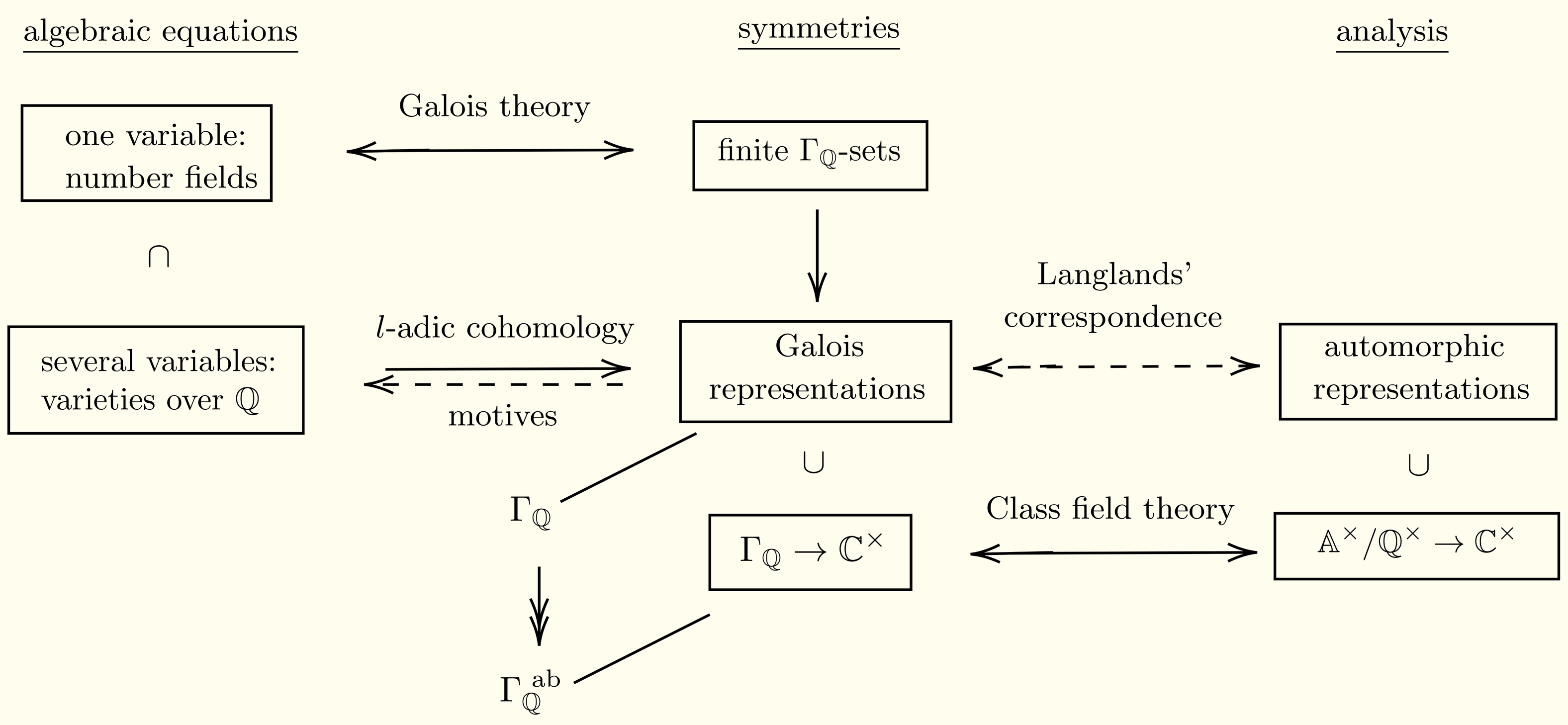


Conjecture (Perversity equals weight)**.**
The perverse filtration induced by the Hitchin integrable system on the cohomology of the moduli space of Higgs bundles agrees with the weight filtration on the cohomology of the character variety under the non-abelian Hodge correspondence:

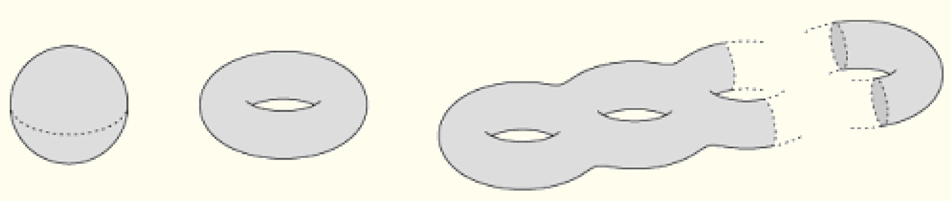
$$P_i(\mathcal{M}_{Dol}) = W_{2i}(\mathcal{M}_B).$$

Arithmetic geometry

Arithmetic geometry is a connection between the discrete world of number theory and continuous world of geometry and analysis. It studies number theoretic questions by recasting them in geometric terms. This allows for the use of the powerful machinery of algebraic geometry.



Arithmetic geometry also connects to physics, mirror symmetry and enumerative geometry via the moduli of curves of genus g, which is the algebraic space parametrizing smooth algebraic curves of genus g.

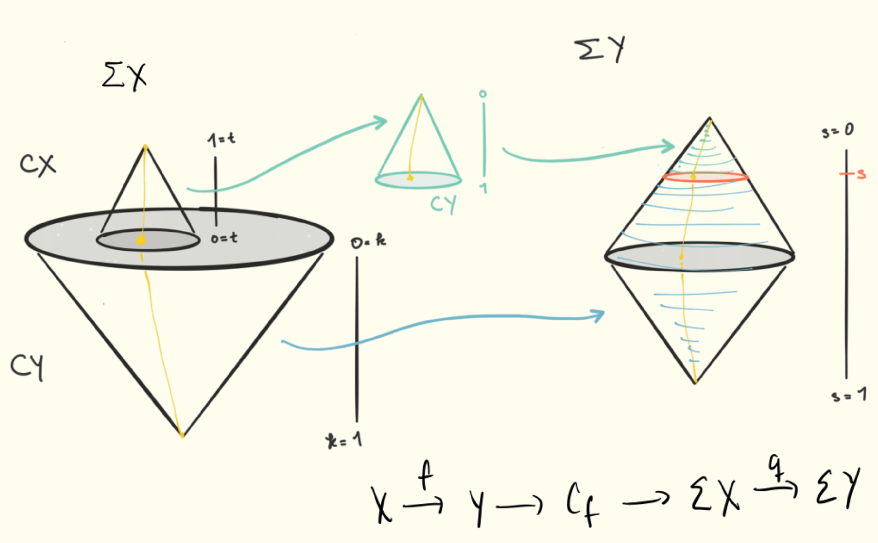


Problem. What is the zeta function of the moduli space of stable curves of genus g over $\mathbb{Z}$? Following Langlands philosophy, every L-function is an L-function of an automorphic representation, so can the L-function of the moduli of stable curves be described in automorphic terms?

Homotopical algebraic geometry

Homotopy theory is the study of shapes, more precisely topological spaces, up to continuous deformations such as stretching, twisting, and bending. Algebraic geometry, on the other hand, is the study of algebraic varieties, solutions to systems of polynomial equations, which are far more rigid than topological spaces. Interestingly, many subtle phenomena in algebraic geometry could be explained using ideas coming from homotopy theory, resulting in numerous breakthroughs during the last decade.

As the most basic example, the *mapping cone* construction is a standard construction in homotopy theory, which, roughly speaking, provides the homotopically correct way to take the difference between two objects. It inspires a similarly named construction in homotopical algebra and thus plays an important role in homotopical algebraic geometry.



Problem. Let G be an algebraic reductive group with maximal torus T, $\mathfrak{g} = \operatorname{Lie} G$, and $\mathfrak{t} = \operatorname{Lie} T$. Show that

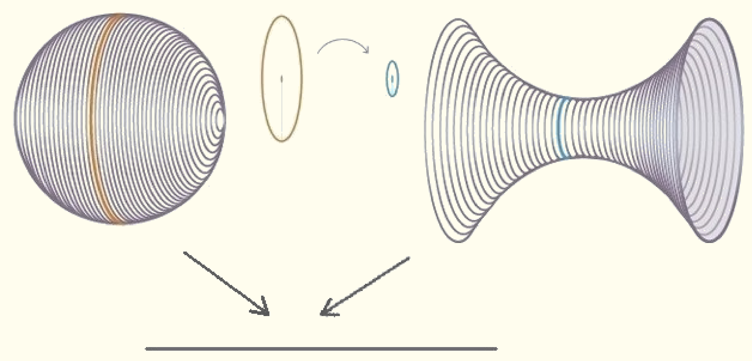
$$\mathcal{O}(T^*(\mathfrak{g}/G)) \simeq \mathcal{O}(\mathfrak{t} \times \mathfrak{t} \times \mathfrak{t}[-1])^W.$$

Mirror symmetry

When studying spaces from differential point of view, one can highlight different aspects of their geometry by equipping the space in question X with different structures. Two examples of this are:

- symplectic structure, when X is thought of as the phase space of a physical system;
- holomorphic structure, when X is cut out by (complex) analytic equations.

It turns out that for certain spaces one can find a mirror partner $X^\vee$, which has these two aspects interchanged. This phenomenon goes under the name *mirror symmetry.*
One can formulate this expectation in terms of equality of numerical invariants, namely Hodge numbers. However, the more profound way is to recast it as an equivalence of certain categories:

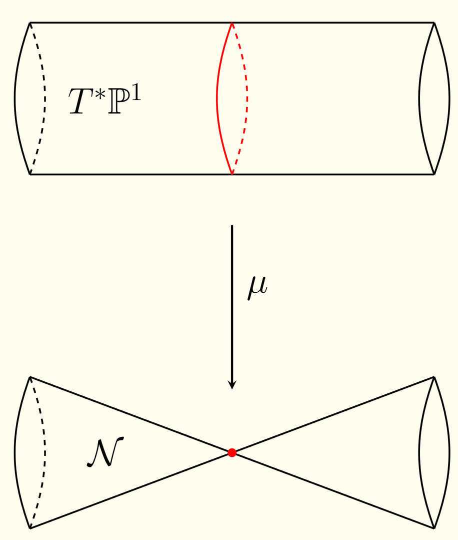


$$D^b(\operatorname{Coh} X) \simeq D^b(\operatorname{Fuk} X^\vee).$$

An example where mirror symmetry is well understood is when X is a torus, and $X^\vee$ is its dual. The SYZ conjecture tells us that all mirror pairs should generically be of this form.
A specific example of SYZ picture is the moduli space of Higgs bundles $\mathcal{M}_G$, where G is a reductive group. Generically, it is a fibration in tori (the *Hitchin fibration*), however singular fibers are notoriously difficult to handle. One can extract some information about those fibers by studying Hecke and Wilson operators on $D^b(\mathcal{M}_G)$. It is known that mirror symmetry exchanges these two families of operators over the smooth locus of Hitchin fibration.

Problem. Show that generic mirror symmetry on $\mathcal{M}_{GL_n}$ extends to an $SL_2(\mathbb{Z})$-equivariant action of an enhanced elliptic Hall algebra.

Geometric representation theory



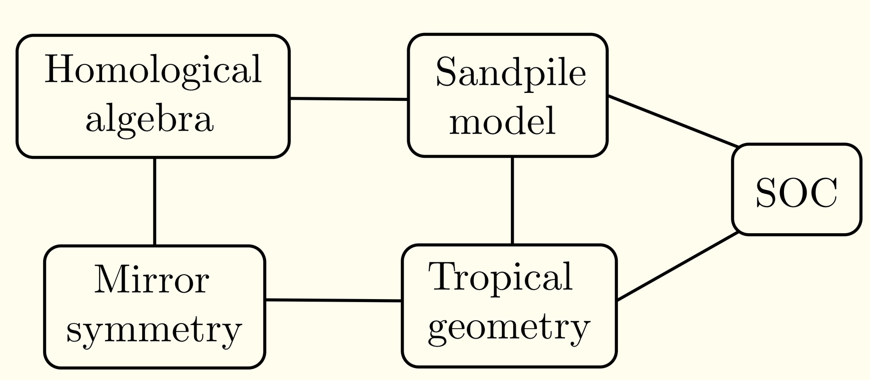
Representation theory is the study of symmetry. The symmetries are encoded by various algebraic objects, such as groups, algebras, Lie algebras and so on, while their study often relies on powerful arguments coming from algebraic geometry. For example, Springer theory provides a geometric realization of representations of the symmetric group S_n, as well as other Weyl groups. Namely, one finds all finite-dimensional representations as homology groups of the fibers of the Springer resolution, which is illustrated on the left for S_2.

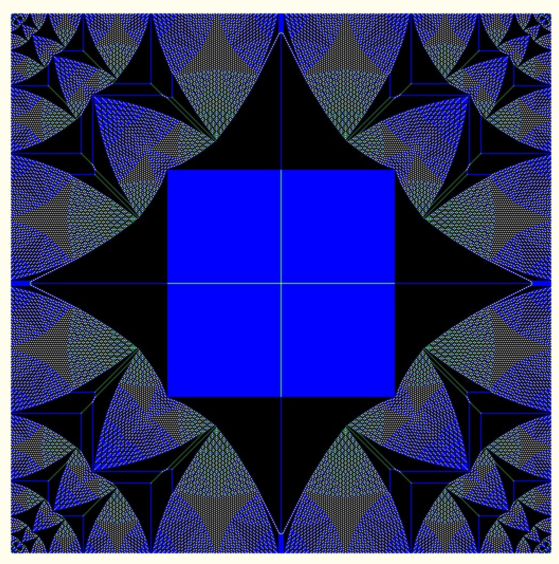
Meanwhile, the Beilinson–Bernstein localization theorem describes representations of reductive Lie algebras in terms of differential operators on the corresponding flag variety. A final example is the geometric Satake theorem, which provides a manifestation of the Langlands dual of a reductive group via perverse sheaves on the corresponding affine Grassmannian.

Problem. What are the symplectic Galois groups acting on homology groups of the fibers of other symplectic resolutions, such as Nakajima quiver varieties?

Sandpile groups

Critical systems are formed at the junction of order and chaos, realizing in some cases a neighborhood of phase transition, and in others demonstrating *self-organizing criticality*. In real life there are plenty of examples: everything about subtle fractality, long distance/time correlations, power-laws and so on. We do not know how to work adequately with them, we cannot predict anything, we don't even understand how to study them. There is a conviction that they possess a universal structure, but no one can say even approximately what this structure is.





There are very few mathematically well-defined SOC systems – the *abelian sandpile model* is historically the first and still the most popular one. It represents an idealized process of gradually increasing pile of sand, where we look at the statistics of the sizes of avalanches induced by adding an extra grain at a random site of a discrete domain.
The model operates on the so-called sandpile group consisting of recurrent piles on the domain. Although it is a finite abelian group, it is usually extremely complicated, and contains a lot of fractals; for instance, the square on the left depicts the neutral element. Recently, we have found an exceptional collection of monomorphisms between sandpile groups on lattice polygons, which allows to take an injective scaling limit.

Problem. Show that the injective scaling limit of sandpile groups is $SL_2(\mathbb{Z})$-invariant.