Anatomy of big algebras

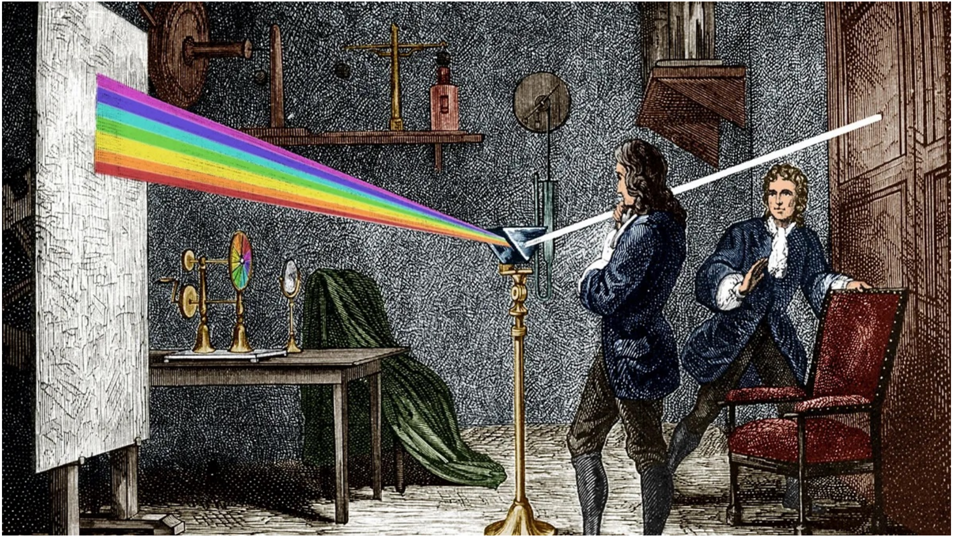
Tamás Hausel

Institute of Science and Technology Austria
http://hausel.ist.ac.at

SMRI Seminar
Sydney Mathematics Research Institute
October 2025

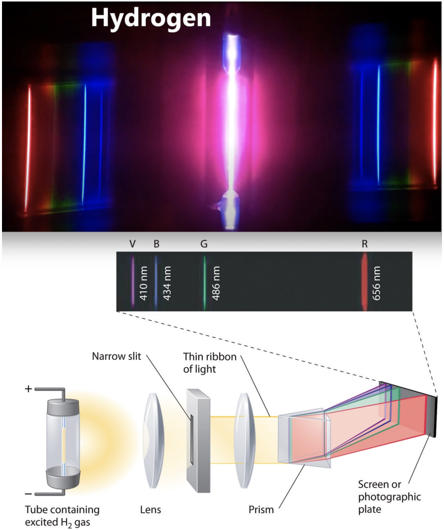
Newton's spectrum

Newton's spectrum



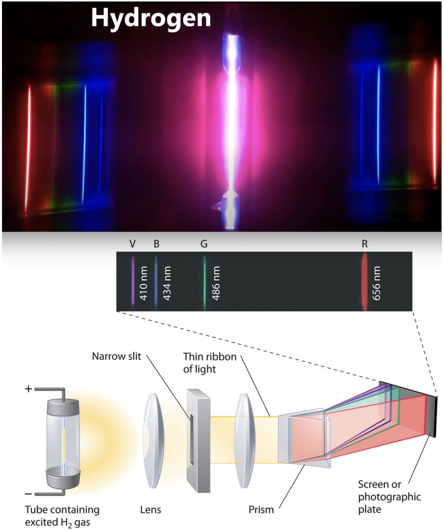
Newton beams white light through prism in Trinity College 1666

Spectrum of hydrogen atom

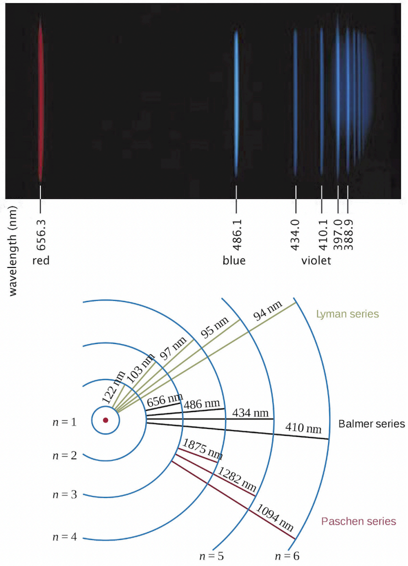


Measuring hydrogen's emmission spectrum

Spectrum of hydrogen atom

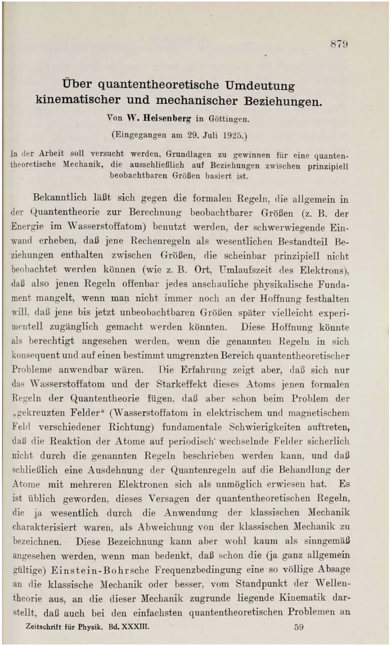


Measuring hydrogen's emmission spectrum



Visible spectrum of hydrogen in Balmer series

Heisenberg’s quantum matrix model



879

Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen.

Von W. Heisenberg in Göttingen.

(Eingegangen am 29. Juli 1925.)

In der Arbeit soll versucht werden, Grundlagen zu gewinnen für eine quantentheoretische Mechanik, die ausschließlich auf Beziehungen zwischen prinzipiell beobachtbaren Größen basiert ist.

Bekanntlich läßt sich gegen die formalen Regeln, die allgemein in der Quantentheorie zur Berechnung beobachtbarer Größen (z. B. der Energie im Wasserstoffatom) benutzt werden, der schwerwiegende Einwand erheben, daß jene Rechenregeln als wesentlichen Bestandteil Beziehungen enthalten zwischen Größen, die scheinbar prinzipiell nicht beobachtet werden können (wie z. B. Ort, Umlaufszeit des Elektrons), daß also jenen Regeln offenbar jedes anschauliche physikalische Fundament mangelt, wenn man nicht immer noch an der Hoffnung festhalten will, daß jene bis jetzt unbeobachtbaren Größen später vielleicht experimentell zugänglich gemacht werden könnten. Diese Hoffnung könnte als berechtigt angesehen werden, wenn die genannten Regeln in sich konsequent und auf einen bestimmt umgrenzten Bereich quantentheoretischer Probleme anwendbar wären. Die Erfahrung zeigt aber, daß sich nur das Wasserstoffatom und der Starkeffekt dieses Atoms jenen formalen Regeln der Quantentheorie fügen, daß aber schon beim Problem der „gekreuzten Felder“ (Wasserstoffatom in elektrischem und magnetischem Feld verschiedener Richtung) fundamentale Schwierigkeiten auftreten, daß die Reaktion der Atome auf periodisch wechselnde Felder sicherlich nicht durch die genannten Regeln beschrieben werden kann, und daß schließlich eine Ausdehnung der Quantenregeln auf die Behandlung der Atome mit mehreren Elektronen sich als unmöglich erwiesen hat. Es ist üblich geworden, dieses Versagen der quantentheoretischen Regeln, die ja wesentlich durch die Anwendung der klassischen Mechanik charakterisiert waren, als Abweichung von der klassischen Mechanik zu bezeichnen. Diese Bezeichnung kann aber wohl kaum als sinngemäß angesehen werden, wenn man bedenkt, daß schon die (ja ganz allgemein gültige) Einstein-Bohrsche Frequenzbedingung eine so völlige Absage an die klassische Mechanik oder besser, vom Standpunkt der Wellentheorie aus, an die dieser Mechanik zugrunde liegende Kinematik darstellt, daß auch bei den einfachsten quantentheoretischen Problemen an

Zeitschrift für Physik. Bd. XXXIII. 59

(Heisenberg 1925)
quantum observables are matrices and their measurements are their discrete eigenvalues

Heisenberg’s quantum matrix model

Received July 29, 1925

12

QUANTUM-THEORETICAL RE-INTERPRETATION OF KINEMATIC AND MECHANICAL RELATIONS

W. HEISENBERG

The present paper seeks to establish a basis for theoretical quantum mechanics founded exclusively upon relationships between quantities which in principle are observable.

It is well known that the formal rules which are used in quantum theory for calculating observable quantities such as the energy of the hydrogen atom may be seriously criticized on the grounds that they contain, as basic element, relationships between quantities that are apparently unobservable in principle, e.g., position and period of revolution of the electron. Thus these rules lack an evident physical foundation, unless one still wants to retain the hope that the hitherto unobservable quantities may later come within the realm of experimental determination. This hope might be regarded as justified if the above-mentioned rules were internally consistent and applicable to a clearly defined range of quantum mechanical problems. Experience however shows that only the hydrogen atom and its Stark effect are amenable to treatment by these formal rules of quantum theory. Fundamental difficulties already arise in the problem of ‘crossed fields’ (hydrogen atom in electric and magnetic fields of differing directions). Also, the reaction of atoms to periodically varying fields cannot be described by these rules. Finally, the extension of the quantum rules to the treatment of atoms having several electrons has proved unfeasible.

It has become the practice to characterize this failure of the quantum-theoretical rules as a deviation from classical mechanics, since the rules themselves were essentially derived from classical mechanics. This characterization has, however, little meaning when one realizes

Editor’s note. This paper was published as Zs. Phys. **33** (1925) 879–893. It was signed ‘Göttingen, Institut für theoretische Physik’.

(Heisenberg 1925) quantum observables are matrices and their measurements are their discrete eigenvalues

Heisenberg's quantum matrix model

- (Heisenberg 1925) quantum observables are matrices

Heisenberg’s quantum matrix model

- (Heisenberg 1925) quantum observables are matrices
- measurement of an observable is an eigenvalue of the matrix

Heisenberg's quantum matrix model

- (Heisenberg 1925) quantum observables are matrices
- measurement of an observable is an eigenvalue of the matrix
  – typically discrete, or even finite, set of possible outcomes

Heisenberg's quantum matrix model

- (Heisenberg 1925) quantum observables are matrices
- measurement of an observable is an eigenvalue of the matrix – typically discrete, or even finite, set of possible outcomes – like the wavelength of the spectral line in the emmission spectrum of hydrogen

Heisenberg's quantum matrix model

- (Heisenberg 1925) quantum observables are matrices
- measurement of an observable is an eigenvalue of the matrix – typically discrete, or even finite, set of possible outcomes – like the wavelength of the spectral line in the emmission spectrum of hydrogen
- (Heisenberg 1927) quantum observables can be measured at the same time only if their matrices commute

Heisenberg's quantum matrix model

- (Heisenberg 1925) quantum observables are matrices
- measurement of an observable is an eigenvalue of the matrix – typically discrete, or even finite, set of possible outcomes – like the wavelength of the spectral line in the emmission spectrum of hydrogen
- (Heisenberg 1927) quantum observables can be measured at the same time only if their matrices commute $\rightsquigarrow$

Heisenberg's quantum matrix model

- (Heisenberg 1925) quantum observables are matrices
- measurement of an observable is an eigenvalue of the matrix – typically discrete, or even finite, set of possible outcomes – like the wavelength of the spectral line in the emmission spectrum of hydrogen
- (Heisenberg 1927) quantum observables can be measured at the same time only if their matrices commute $\rightsquigarrow$ *uncertainty principle* for non-commuting observables

Heisenberg's quantum matrix model

- (Heisenberg 1925) quantum observables are matrices
- measurement of an observable is an eigenvalue of the matrix – typically discrete, or even finite, set of possible outcomes – like the wavelength of the spectral line in the emmission spectrum of hydrogen
- (Heisenberg 1927) quantum observables can be measured at the same time only if their matrices commute $\rightsquigarrow$ *uncertainty principle* for non-commuting observables like position and momentum

Heisenberg's quantum matrix model

- (Heisenberg 1925) quantum observables are matrices
- measurement of an observable is an eigenvalue of the matrix – typically discrete, or even finite, set of possible outcomes – like the wavelength of the spectral line in the emmission spectrum of hydrogen
- (Heisenberg 1927) quantum observables can be measured at the same time only if their matrices commute $\rightsquigarrow$ *uncertainty principle* for non-commuting observables like position and momentum
- matrix algebra is cumbersome because of non-commutativity

Heisenberg's quantum matrix model

- (Heisenberg 1925) quantum observables are matrices
- measurement of an observable is an eigenvalue of the matrix – typically discrete, or even finite, set of possible outcomes – like the wavelength of the spectral line in the emmission spectrum of hydrogen
- (Heisenberg 1927) quantum observables can be measured at the same time only if their matrices commute $\rightsquigarrow$ *uncertainty principle* for non-commuting observables like position and momentum
- matrix algebra is cumbersome because of non-commutativity
- (Schrödinger 1928) "I naturally knew about his theory, but was discouraged, if not repelled, by what appeared to me as very difficult methods of transcendental algebra, and by the want of Anschaulichkeit."

Schrödinger's quantum wave model

Schrödinger's quantum wave model

- Anschaulichkeit = intelligibility + visualisability

Schrödinger's quantum wave model

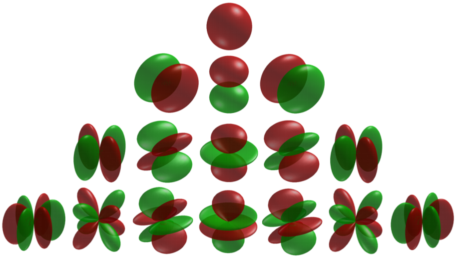
- Anschaulichkeit = intelligibility + visualisability
- (Schrödinger 1926) quantum particles can be modelled by waves

Schrödinger's quantum wave model

- Anschaulichkeit = intelligibility + visualisability
- (Schrödinger 1926) quantum particles can be modelled by waves, solving Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$:

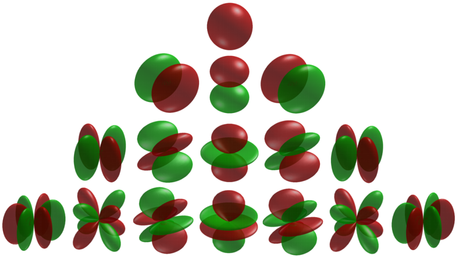
Schrödinger's quantum wave model

- Anschaulichkeit = intelligibility + visualisability
- (Schrödinger 1926) quantum particles can be modelled by waves, solving Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$:



Schrödinger's quantum wave model

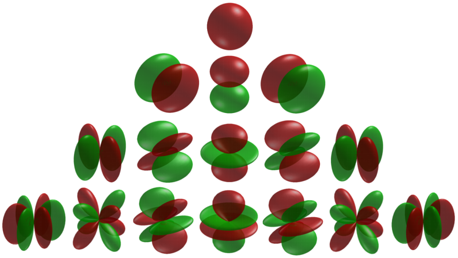
- Anschaulichkeit = intelligibility + visualisability
- (Schrödinger 1926) quantum particles can be modelled by waves, solving Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$:



- (Heisenberg 1926) "The more I think of the physical part of Schrödinger's theory, the more abominable I find it. What Schrödinger writes about Anschaulichkeit makes scarcely any sense, in other words I think it is crap (=Mist)."

Schrödinger's quantum wave model

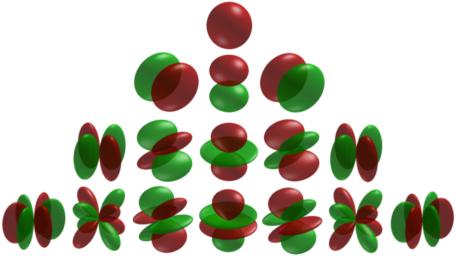
- Anschaulichkeit = intelligibility + visualisability
- (Schrödinger 1926) quantum particles can be modelled by waves, solving Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$:



- (Heisenberg 1926) "The more I think of the physical part of Schrödinger's theory, the more abominable I find it. What Schrödinger writes about Anschaulichkeit makes scarcely any sense, in other words I think it is crap (=Mist)."
- (Schrödinger 1926, Dirac 1926, von Neumann 1929) Heisenberg picture = Schrödinger picture

Schrödinger's quantum wave model

- Anschaulichkeit = intelligibility + visualisability
- (Schrödinger 1926) quantum particles can be modelled by waves, solving Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$:

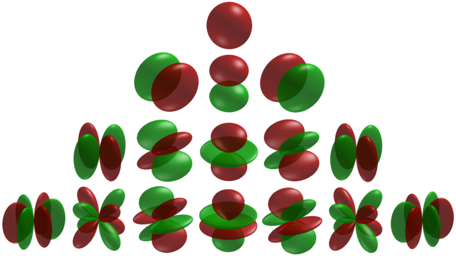


- (Heisenberg 1926) "The more I think of the physical part of Schrödinger's theory, the more abominable I find it. What Schrödinger writes about Anschaulichkeit makes scarcely any sense, in other words I think it is crap (=Mist)."
- (Schrödinger 1926, Dirac 1926, von Neumann 1929) Heisenberg picture = Schrödinger picture

$$\ell^2(\mathbb{N}) \cong L^2(\mathbb{R})$$

Schrödinger's quantum wave model

- Anschaulichkeit = intelligibility + visualisability
- (Schrödinger 1926) quantum particles can be modelled by waves, solving Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$:



- (Heisenberg 1926) "The more I think of the physical part of Schrödinger's theory, the more abominable I find it. What Schrödinger writes about Anschaulichkeit makes scarcely any sense, in other words I think it is crap (=Mist)."
- (Schrödinger 1926, Dirac 1926, von Neumann 1929) Heisenberg picture = Schrödinger picture

$$\ell^2(\mathbb{N}) \cong L^2(\mathbb{R}) \text{ (Riesz-Fischer 1907)}$$

Schrödinger’s quantum wave model

3. ***Quantisierung als Eigenwertproblem;***
von E. Schrödinger.

(Erste Mitteilung.)

§ 1. In dieser Mitteilung möchte ich zunächst an dem einfachsten Fall des (nichtrelativistischen und ungestörten) Wasserstoffatoms zeigen, daß die übliche Quantisierungsvorschrift sich durch eine andere Forderung ersetzen läßt, in der kein Wort von „ganzen Zahlen“ mehr vorkommt. Vielmehr ergibt sich die Ganzzahligkeit auf dieselbe natürliche Art, wie etwa die Ganzzahligkeit der *Knotenzahl* einer schwingenden Saite. Die neue Auffassung ist verallgemeinerungsfähig und rührt, wie ich glaube, sehr tief an das wahre Wesen der Quantenvorschriften.

Die übliche Form der letzteren knüpft an die Hamiltonsche partielle Differentialgleichung an:

$$H\left(q, \frac{\partial S}{\partial q}\right) = E\,. \tag{1}$$

Es wird von dieser Gleichung eine Lösung gesucht, welche sich darstellt als *Summe* von Funktionen je einer einzigen der unabhängigen Variablen q.

Wir führen nun für S eine neue unbekannte ψ ein derart, daß ψ als ein *Produkt* von eingriffigen Funktionen der einzelnen Koordinaten erscheinen würde. D. h. wir setzen

$$S = K \lg \psi\,. \tag{2}$$

Die Konstante K muß aus dimensionellen Gründen eingeführt werden, sie hat die Dimension einer *Wirkung*. Damit erhält man

$$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) = E\,. \tag{1'}$$

Wir suchen nun *nicht* eine Lösung der Gleichung (1′), sondern wir stellen folgende Forderung. Gleichung (1′) läßt sich bei Vernachlässigung der Massenveränderlichkeit stets, bei Berücksichtigung derselben wenigstens dann, wenn es sich um das *Ein*elektronenproblem handelt, auf die Gestalt bringen: quadratische

(Schrödinger 1926)
quantum particles are waves which are eigenvectors of the Hamiltonian operator

Schrödinger’s quantum wave model

3. Quantisation as an eigenvalue problem; by E. Schrödinger*

(first communication.)

§ 1. In this communication I would like first to show, in the simplest case of the (non-relativistic and unperturbed) hydrogen atom, that the usual prescription for quantisation can be substituted by another requirement in which no word about “integer numbers” occurs anymore. Rather, the integerness[1] emerges in the same natural way as, for example, the integerness of the *number of knots* of a vibrating string. The new interpretation is generalisable and touches, as I believe, very deeply the true essence of the quantisation prescription.

The usual form of the latter is tied to the Hamiltonian partial differential equation:

$$H\left(q, \frac{\partial S}{\partial q}\right) = E\,. \tag{1}$$

It is looked for a solution of this equation that appears as a *sum* of functions, each of only one of the independent variables q.

We introduce now in place of S a new, unknown function ψ in such a manner that ψ would appear as a *product* of suitable functions of the single coordinates. That is, we set:

$$S = K \lg \psi\,. \tag{2}$$

The constant K must be introduced for dimensional reasons and it has the dimension of an *action*. With this one obtains:

$$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) = E\,. \tag{1'}$$

We *do not* look now for a solution of equation (1′), but we stipulate the following requirement. Neglecting the variability of the masses, or considering it at least as long as the *single* electron problem is concerned, equation (1′) can always be brought to the form: a quadratic form for ψ and its first derivatives $= 0$. We look for such

*Original title: *Quantisierung als Eigenwertproblem*. Published in: *Annalen der Physik* 79 (1926): 361-376. Translated by Oliver F. Piattella. E-mail: oliver.piattella@cosmo-ufes.org

(Schrödinger 1926)
quantum particles are waves which are eigenvectors of the Hamiltonian operator

Hilbert's spectrum

Hilbert's spectrum

- $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ *matrix*: grid of numbers
  $$\vec{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$ *vector*: column of numbers
  $$A\vec{b} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$ matrix-vector product

Hilbert's spectrum

- $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad Ab = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

  matrix: grid of numbers — *vector*: column of numbers — matrix-vector product

- matrix transformation: $$A : \begin{array}{ccc} \mathbb{C}^3 & \rightarrow & \mathbb{C}^3 \\ b & \mapsto & Ab \end{array}$$

Hilbert's spectrum

- $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad A\vec{b} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

  matrix: grid of numbers — *vector*: column of numbers — matrix-vector product
- matrix transformation: $$\begin{array}{rccc} A: & \mathbb{C}^3 & \to & \mathbb{C}^3 \\ & b & \mapsto & Ab \end{array}$$
- when $Ab = \lambda b$, then b is an eigenvector of A of eigenvalue λ

Hilbert's spectrum

- $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad Ab = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

  matrix: grid of numbers — *vector*: column of numbers — matrix-vector product

- matrix transformation: $$\begin{array}{rccc} A: & \mathbb{C}^3 & \to & \mathbb{C}^3 \\ & b & \mapsto & Ab \end{array}$$
- when $Ab = \lambda b$, then b is an eigenvector of A of eigenvalue λ
  spectrum of A is the set of all eigenvalues of A

Hilbert's spectrum

- $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad Ab = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

  matrix: grid of numbers — *vector*: column of numbers — matrix-vector product
- matrix transformation: $$\begin{array}{rccc} A: & \mathbb{C}^3 & \to & \mathbb{C}^3 \\ & b & \mapsto & Ab \end{array}$$
- when $Ab = \lambda b$, then b is an eigenvector of A of eigenvalue λ

  spectrum of A is the set of all eigenvalues of A
- Example: Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$

Hilbert's spectrum

- $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad Ab = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

  matrix: grid of numbers — *vector*: column of numbers — matrix-vector product

- matrix transformation: $$A : \begin{array}{ccc} \mathbb{C}^3 & \to & \mathbb{C}^3 \\ b & \mapsto & Ab \end{array}$$
- when $Ab = \lambda b$, then b is an eigenvector of A of eigenvalue λ
  spectrum of A is the set of all eigenvalues of A
- Example: Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$;
  spectrum of $\hat{H}$ are the possible energy values E

Hilbert's spectrum

- $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad A\vec{b} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

  matrix: grid of numbers — *vector*: column of numbers — matrix-vector product

- matrix transformation: $$\begin{array}{rccc} A: & \mathbb{C}^3 & \to & \mathbb{C}^3 \\ & b & \mapsto & Ab \end{array}$$
- when $Ab = \lambda b$, then b is an eigenvector of A of eigenvalue λ
  spectrum of A is the set of all eigenvalues of A
- Example: Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$;
  spectrum of $\hat{H}$ are the possible energy values E
- matrix multiplication non-commutative

Hilbert's spectrum

- $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad Ab = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

  matrix: grid of numbers — *vector*: column of numbers — matrix-vector product

- matrix transformation: $\begin{array}{cccc} A: & \mathbb{C}^3 & \to & \mathbb{C}^3 \\ & b & \mapsto & Ab \end{array}$
- when $Ab = \lambda b$, then b is an eigenvector of A of eigenvalue λ
  spectrum of A is the set of all eigenvalues of A
- Example: Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$;
  spectrum of $\hat{H}$ are the possible energy values E
- matrix multiplication non-commutative:

$$Q_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

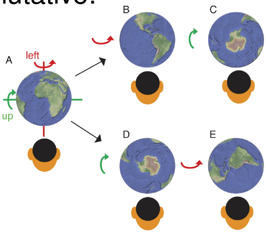
$$Q_1Q_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \qquad Q_2Q_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

- $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad A\vec{b} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

  matrix: grid of numbers — *vector*: column of numbers — matrix-vector product
- matrix transformation: $$A : \begin{array}{ccc} \mathbb{C}^3 & \to & \mathbb{C}^3 \\ b & \mapsto & Ab \end{array}$$
- when $Ab = \lambda b$, then b is an eigenvector of A of eigenvalue λ
  spectrum of A is the set of all eigenvalues of A
- Example: Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$;
  spectrum of $\hat{H}$ are the possible energy values E
- matrix multiplication non-commutative:

$$Q_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$Q_1 Q_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \qquad Q_2 Q_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$



Hilbert's spectrum

METHODEN
DER MATHEMATISCHEN
PHYSIK

VON

R. COURANT
ORD. PROFESSOR DER MATHEMATIK
AN DER UNIVERSITÄT GÖTTINGEN

UND

D. HILBERT
GEH. REG.-RAT · ORD. PROFESSOR DER
MATHEMATIK AN DER UNIVERSITÄT
GÖTTINGEN

ERSTER BAND

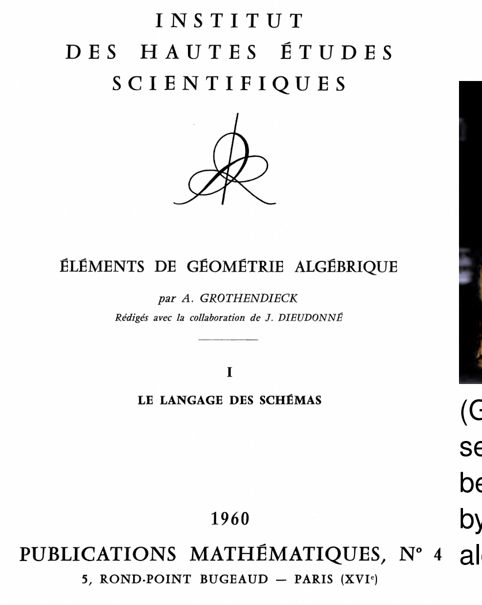
MIT 29 ABBILDUNGEN

BERLIN
VERLAG VON JULIUS SPRINGER
1924

(Hilbert 1924) "I developed my theory of infinitely many variables from purely mathematical interests, and even called it 'spectral analysis' without any presentiment that it would later find an application to the actual spectrum of physics."

Grothendieck's spectrum

Grothendieck’s spectrum



INSTITUT
DES HAUTES ÉTUDES
SCIENTIFIQUES

ÉLÉMENTS DE GÉOMÉTRIE ALGÉBRIQUE

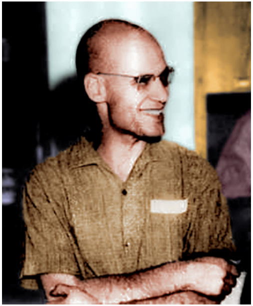
par A. GROTHENDIECK
Rédigés avec la collaboration de J. DIEUDONNÉ

I

LE LANGAGE DES SCHÉMAS

1960

PUBLICATIONS MATHÉMATIQUES, N° 4
5, ROND-POINT BUGEAUD — PARIS (XVIe)



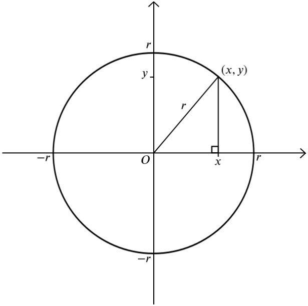
(Grothendieck 1960)
sets up a two-way dictionary
between algebra and geometry
by visualising a commutative
algebra via its spectrum

Grothendieck's spectrum

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra

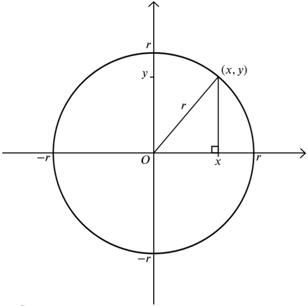
- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra



circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra

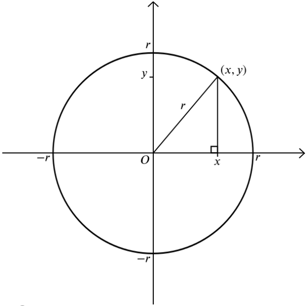


circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra

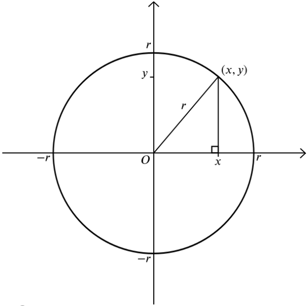


circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra

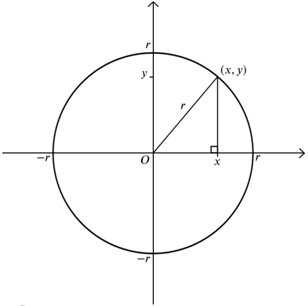


circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R = \mathbb{R}[x, y]$ polynomial functions on (x, y)-plane

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra



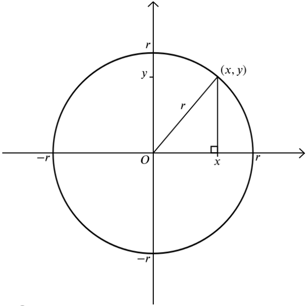
circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R = \mathbb{R}[x, y]$ polynomial functions on (x, y)-plane;

  the point (x_0, y_0)

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra

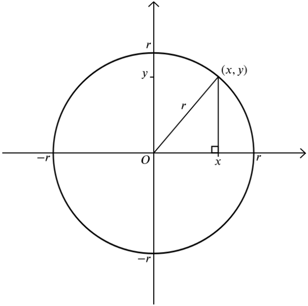


circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R = \mathbb{R}[x, y]$ polynomial functions on (x, y)-plane; the point (x_0, y_0) corresponds to
$$\begin{array}{cccc} f : & R & \to & \mathbb{R} \\ & p & \mapsto & p(x_0, y_0) \end{array}$$

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra

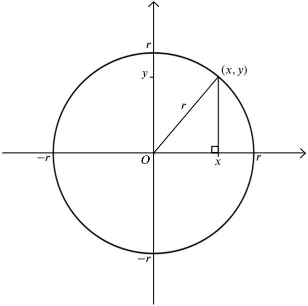


circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R = \mathbb{R}[x, y]/(x^2 + y^2 - r^2)$

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra

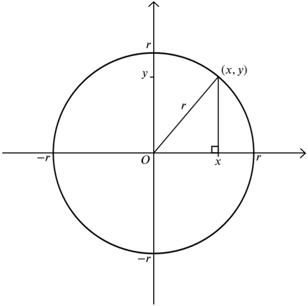


circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R = \mathbb{R}[x, y]/(x^2 + y^2 - r^2)$; for solution $x_0^2 + y_0^2 = r^2$

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra



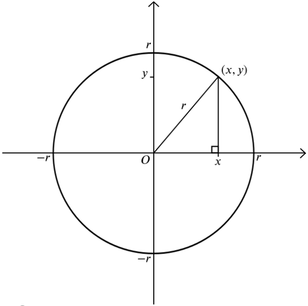
circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R = \mathbb{R}[x, y]/(x^2 + y^2 - r^2)$; for solution $x_0^2 + y_0^2 = r^2$

  (x_0, y_0) corresponds to $\begin{array}{cccc} f : & R & \to & \mathbb{R} \\ & p & \mapsto & p(x_0, y_0) \end{array}$

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra

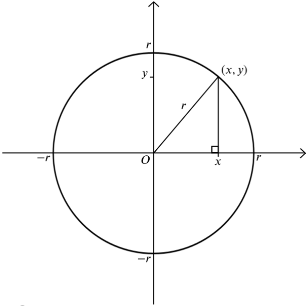


circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R \subset M_{nxn}(\mathbb{C})$ <u>commutative</u> algebra of $n \times n$ matrices

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra



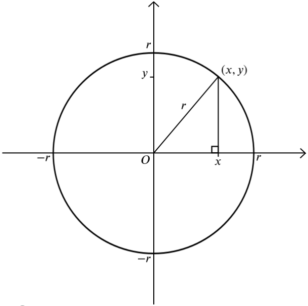
circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R \subset M_{nxn}(\mathbb{C})$ commutative algebra of $n \times n$ matrices;

  common eigenvector b

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra



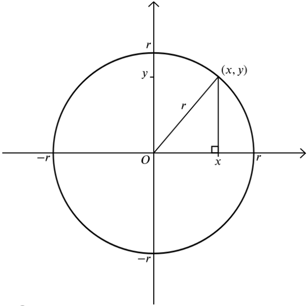
circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R \subset M_{nxn}(\mathbb{C})$ <u>commutative</u> algebra of $n \times n$ matrices;

  common eigenvector b; $Ab = \lambda_A b$

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra



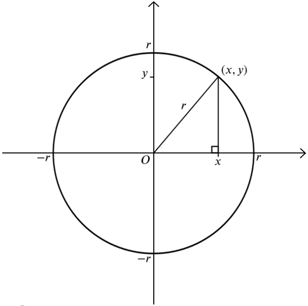
circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R \subset M_{nxn}(\mathbb{C})$ <u>commutative</u> algebra of $n \times n$ matrices;

common eigenvector b; $Ab = \lambda_A b \rightsquigarrow$ $\begin{array}{rccc} f: & R & \to & \mathbb{C} \\ & A & \mapsto & \lambda_A \end{array}$

Grothendieck's spectrum

- (Descartes 1637) $\rightsquigarrow$ coordinates solve geometry with algebra



circle of radius r is set of points (x, y) on the plane solving $x^2 + y^2 = r^2$

- (Groethendieck 1960) encodes the circle by the commutative algebra of functions on it: $\mathbb{R}[x, y]/(x^2 + y^2 - r^2)$
- *spectrum* of a commutative algebra R is the space of multiplicative functions $f : R \to \mathbb{R}$
- Example: $R \subset M_{nxn}(\mathbb{C})$ <u>commutative</u> algebra of $n \times n$ matrices; common eigenvector b; $Ab = \lambda_A b \rightsquigarrow$ $\begin{array}{rccc} f: & R & \to & \mathbb{C} \\ & A & \mapsto & \lambda_A \end{array}$

  in this case: Grothendieck's spectrum = Hilbert's spectrum

Big algebra

Big algebra

PNAS RESEARCH ARTICLE | MATHEMATICS OPEN ACCESS

Check for updates

Commutative avatars of representations of semisimple Lie groups

Tamás Hausel[a,1]

Edited by Kenneth Ribet, University of California, Berkeley, CA; received November 7, 2023; accepted July 23, 2024

Here we announce the construction and properties of a big commutative subalgebra of the Kirillov algebra attached to a finite dimensional irreducible representation of a complex semisimple Lie group. They are commutative finite flat algebras over the cohomology of the classifying space of the group. They are isomorphic with the equivariant intersection cohomology of affine Schubert varieties, endowing the latter with a new ring structure. Study of the finer aspects of the structure of the big algebras will also furnish the stalks of the intersection cohomology with ring structure, thus ringifying Lusztig's *q*-weight multiplicity polynomials i.e., certain affine Kazhdan–Lusztig polynomials.

representations of Lie groups | Hitchin integrable system | Higgs field | equivariant cohomology | intersection cohomology

1. Kirillov and Medium Algebras

Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, which we

Significance

Representations of continuous symmetry groups by matrices are fundamental to mathematical models of quantum physics and also to the Langlands program in number theory. Here, we attach a commutative matrix algebra, called big algebra, to a noncommutative irreducible matrix representation of a bounded continuous symmetry group. We show that the

Big algebra

Big algebra

- a multiplicative $\rho : \mathrm{G} \rightarrow \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n \times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^{\rho} \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathscr{B}^{\rho} \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathscr{B}^{\rho} \subset \mathit{Maps}(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := \mathit{Lie}(\mathrm{G})$

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^{\rho} \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{cccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{rccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$
- (Hausel 2024) $\rightsquigarrow$ wonderful properties of $\mathcal{B}^\rho$:

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{cccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$
- (Hausel 2024) $\rightsquigarrow$ wonderful properties of $\mathcal{B}^\rho$:
  1. it is a maximal <u>commutative</u> subalgebra

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{rccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$
- (Hausel 2024) $\rightsquigarrow$ wonderful properties of $\mathcal{B}^\rho$:
  1. it is a maximal commutative subalgebra
  2. free of rank n over polynomial algebra $\mathbb{C}[\mathfrak{g}]^{\mathrm{G}} = \mathbb{C}[\mathfrak{h}]^W$

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{cccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$
- (Hausel 2024) $\rightsquigarrow$ wonderful properties of $\mathcal{B}^\rho$:
  1. it is a maximal <u>commutative</u> subalgebra
  2. free of rank n over polynomial algebra $\mathbb{C}[\mathfrak{g}]^{\mathrm{G}} = \mathbb{C}[\mathfrak{h}]^W$
  3. $\mathcal{B}^\rho \cong IH^*_{\mathrm{G}^\vee}(\mathrm{Gr}_\rho)$

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{rccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$
- (Hausel 2024) $\rightsquigarrow$ wonderful properties of $\mathcal{B}^\rho$:
  1. it is a maximal commutative subalgebra
  2. free of rank n over polynomial algebra $\mathbb{C}[\mathfrak{g}]^{\mathrm{G}} = \mathbb{C}[\mathfrak{h}]^W$
  3. $\mathcal{B}^\rho \cong IH^*_{\mathrm{G}^\vee}(\mathrm{Gr}_\rho)$ equivariant intersection cohomology of affine Schubert variety for Langlands dual $\mathrm{G}^\vee$

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{cccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$
- (Hausel 2024) $\rightsquigarrow$ wonderful properties of $\mathcal{B}^\rho$:
  1. it is a maximal commutative subalgebra
  2. free of rank n over polynomial algebra $\mathbb{C}[\mathfrak{g}]^{\mathrm{G}} = \mathbb{C}[\mathfrak{h}]^W$
  3. $\mathcal{B}^\rho \cong IH^*_{\mathrm{G}^\vee}(\mathrm{Gr}_\rho)$ equivariant intersection cohomology of affine Schubert variety for Langlands dual $\mathrm{G}^\vee$
  4. $\mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(\mathbb{C}[\mathfrak{h}]^{\mathrm{W}})$

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{rccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$
- (Hausel 2024) $\rightsquigarrow$ wonderful properties of $\mathcal{B}^\rho$:
  1. it is a maximal commutative subalgebra
  2. free of rank n over polynomial algebra $\mathbb{C}[\mathfrak{g}]^{\mathrm{G}} = \mathbb{C}[\mathfrak{h}]^W$
  3. $\mathcal{B}^\rho \cong IH^*_{\mathrm{G}^\vee}(\mathrm{Gr}_\rho)$ equivariant intersection cohomology of affine Schubert variety for Langlands dual $\mathrm{G}^\vee$
  4. $\mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(\mathbb{C}[\mathfrak{h}]^{\mathrm{W}}) \cong \mathfrak{h}//W$ models Hitchin system on certain Lagrangians

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{rccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$
- (Hausel 2024) $\rightsquigarrow$ wonderful properties of $\mathcal{B}^\rho$:
  1. it is a maximal <u>commutative</u> subalgebra
  2. free of rank n over polynomial algebra $\mathbb{C}[\mathfrak{g}]^{\mathrm{G}} = \mathbb{C}[\mathfrak{h}]^W$
  3. $\mathcal{B}^\rho \cong IH^*_{\mathrm{G}^\vee}(\mathrm{Gr}_\rho)$ equivariant intersection cohomology of affine Schubert variety for Langlands dual $\mathrm{G}^\vee$
  4. $\mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(\mathbb{C}[\mathfrak{h}]^{\mathrm{W}}) \cong \mathfrak{h}/\!/W$ models Hitchin system on certain Lagrangians
  5. constructed using (Feigin–Frenkel 1992) center of certain vertex algebra

Spectrum of standard representation of $SO(3)$

Spectrum of standard representation of $SO(3)$

- $SO(3)$ symmetry group of sphere

Spectrum of standard representation of $\mathrm{SO}(3)$

- $\mathrm{SO}(3)$ symmetry group of sphere
- $\rho : \mathrm{SO}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation

Spectrum of standard representation of $\mathrm{SO}(3)$

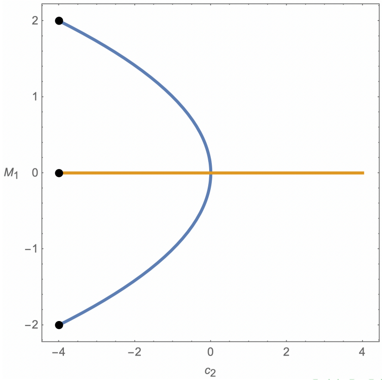
- $\mathrm{SO}(3)$ symmetry group of sphere
- $\rho : \mathrm{SO}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra
  $\mathcal{B}^\rho = \mathbb{C}[c_2, M_1]/(M_1^3 + c_2 M_1) = \mathbb{C}[c_2, M_1]/(M_1(M_1^2 + c_2))$

Spectrum of standard representation of $\mathrm{SO}(3)$

- $\mathrm{SO}(3)$ symmetry group of sphere
- $\rho : \mathrm{SO}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra
  $\mathcal{B}^\rho = \mathbb{C}[c_2, M_1]/(M_1^3 + c_2 M_1) = \mathbb{C}[c_2, M_1]/(M_1(M_1^2 + c_2))$
  by Cayley-Hamilton

Spectrum of standard representation of $\mathrm{SO}(3)$

- $\mathrm{SO}(3)$ symmetry group of sphere
- $\rho : \mathrm{SO}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra
  $\mathcal{B}^\rho = \mathbb{C}[c_2, M_1]/(M_1^3 + c_2 M_1) = \mathbb{C}[c_2, M_1]/(M_1(M_1^2 + c_2))$
  by Cayley-Hamilton
- its spectrum with *principal spectrum* at $c_2 = -4$:



Spectrum of standard representation of $SU(3)$

Spectrum of standard representation of $SU(3)$

- $SU(3)$ special unitary group

Spectrum of standard representation of $SU(3)$

- $SU(3)$ special unitary group

Spectrum of standard representation of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{3\times3}(\mathbb{C})$ standard representation

Spectrum of standard representation of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra is $\mathbb{C}[c_2, c_3, M_1]/(M_1^3 + c_2 M_1 + c_3)$

Spectrum of standard representation of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra is $\mathbb{C}[c_2, c_3, M_1]/(M_1^3 + c_2 M_1 + c_3)$ by Cayley-Hamilton

Spectrum of standard representation of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra is $\mathbb{C}[c_2, c_3, M_1]/(M_1^3 + c_2 M_1 + c_3)$ by Cayley-Hamilton
- its spectrum

Spectrum of standard representation of $\mathrm{SU}(3)$

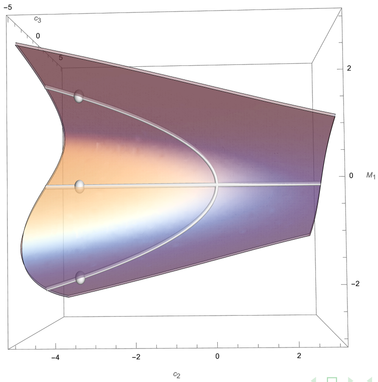
- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra is $\mathbb{C}[c_2, c_3, M_1]/(M_1^3 + c_2 M_1 + c_3)$ by Cayley-Hamilton
- its spectrum with *skeleton* at $c_3 = 0$

Spectrum of standard representation of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra is $\mathbb{C}[c_2, c_3, M_1]/(M_1^3 + c_2 M_1 + c_3)$ by Cayley-Hamilton
- its spectrum with *skeleton* at $c_3 = 0$ and *principal spectrum* at $c_2 = -4$

Spectrum of standard representation of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra is $\mathbb{C}[c_2, c_3, M_1]/(M_1^3 + c_2 M_1 + c_3)$ by Cayley-Hamilton
- its spectrum with *skeleton* at $c_3 = 0$ and *principal spectrum* at $c_2 = -4$:



Spectrum of third symmetric power of $SU(3)$

Spectrum of third symmetric power of $SU(3)$

- $SU(3)$ special unitary group

Spectrum of third symmetric power of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{10\times 10}(\mathbb{C}) = M(S^3(\mathbb{C}^3))$ third symmetric power

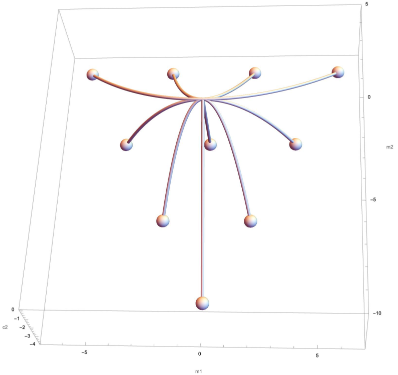
Spectrum of third symmetric power of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{10\times 10}(\mathbb{C}) = M(S^3(\mathbb{C}^3))$ third symmetric power
- its big algebra

$$\mathrm{C}[c_2, c_3, M_1, M_2] / \left(\begin{array}{l} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{array} \right)$$

Spectrum of third symmetric power of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{10\times 10}(\mathbb{C}) = M(S^3(\mathbb{C}^3))$ third symmetric power
- its big algebra
  $\mathrm{C}[c_2, c_3, M_1, M_2] / \left(\begin{array}{l} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{array} \right)$
- its skeleton at $c_3 = 0$ and principal spectrum at $c_2 = -4$:



Gell-Mann's spectrum

Gell-Mann’s spectrum

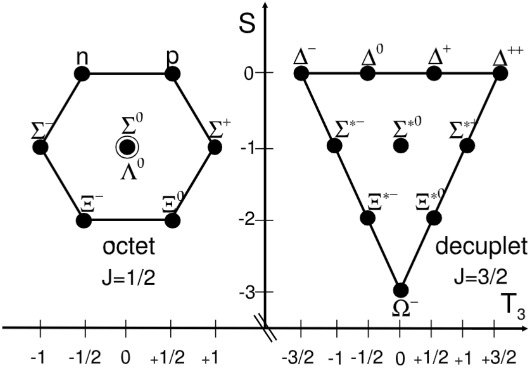
- experiments in the 1950’s produced a *particle zoo* of hundreds of different hadrons

Gell-Mann's spectrum

- experiments in the 1950's produced a *particle zoo* of hundreds of different hadrons
- Gell-Mann 1960s organized them into octets and decuplets

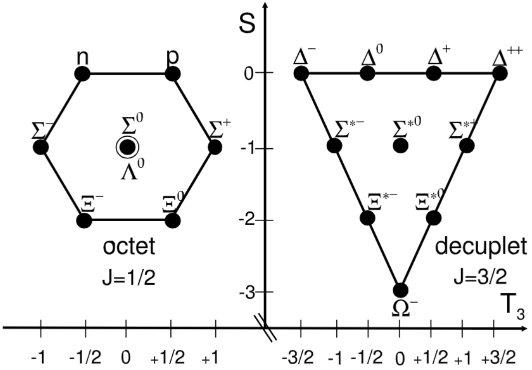
Gell-Mann's spectrum

- experiments in the 1950's produced a *particle zoo* of hundreds of different hadrons
- Gell-Mann 1960s organized them into octets and decuplets



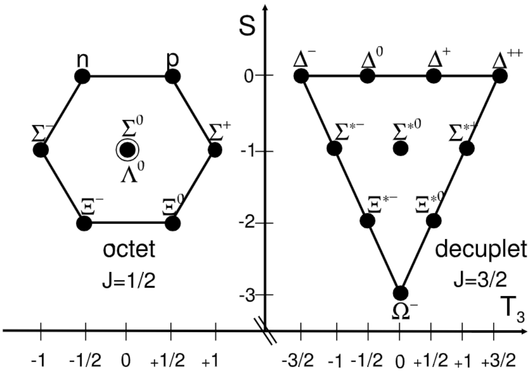
Gell-Mann's spectrum

- experiments in the 1950's produced a *particle zoo* of hundreds of different hadrons
- Gell-Mann 1960s organized them into octets and decuplets



Gell-Mann's spectrum

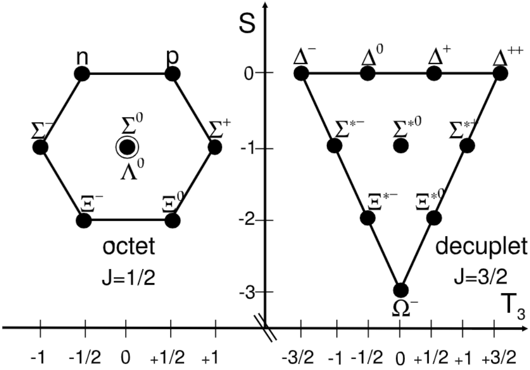
- experiments in the 1950's produced a *particle zoo* of hundreds of different hadrons
- Gell-Mann 1960s organized them into octets and decuplets



- he argued that they are composite particles

Gell-Mann's spectrum

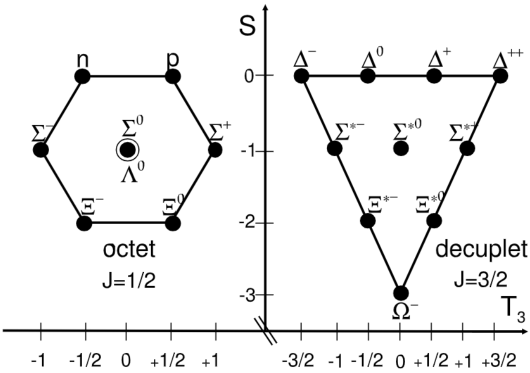
- experiments in the 1950's produced a *particle zoo* of hundreds of different hadrons
- Gell-Mann 1960s organized them into octets and decuplets



- he argued that they are composite particles made up of quarks and antiquarks

Gell-Mann's spectrum

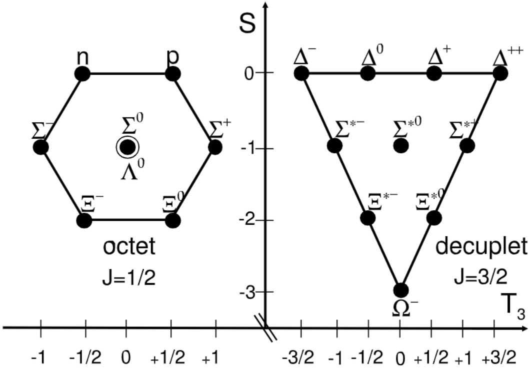
- experiments in the 1950's produced a *particle zoo* of hundreds of different hadrons
- Gell-Mann 1960s organized them into octets and decuplets



- he argued that they are composite particles made up of quarks and antiquarks, which correspond to the fundamental representations of $\mathrm{SU}(3)$

Gell-Mann's spectrum

- experiments in the 1950's produced a *particle zoo* of hundreds of different hadrons
- Gell-Mann 1960s organized them into octets and decuplets



- he argued that they are composite particles made up of quarks and antiquarks, which correspond to the fundamental representations of $\mathrm{SU}(3) \rightsquigarrow$ Quantum-Chromo-Dynamics

Gell-Mann's spectrum

From: *Proc Intern Conf High Energy Phys* (CERN, 1962), p. 805

Strange Particle Physics. Strong Interactions

M. Gell-Mann

SNOW: I know that at 1.5 GeV in the CERN K^-p experiment in the fourbody reaction $\Sigma^{\pm}\pi^{\mp}\pi^-\pi^+$ one looks for $T = 2$ resonances, but when one looks at these reactions the dominant things that one sees are $T = 0$ resonances. I think the Berkeley data which has more statistical weight shows the same thing: they do not see any evidence for this.

SAKURAI: It is now plausible that the 1405 MeV Y_0^* but not the 1385 MeV Y_1^* is an s-wave $\bar{K}N$ bound state of the Dalitz-Tuan type. So it is worth asking what kind of dynamical mechanism is responsible for binding a $\bar{K}$ and an N in the $T = 0$ state. Along this line a student of mine, Richard Arnold at Chicago, has performed a very crude N/D calculation to show that for reasonable values of coupling constants the forces due to the exchanges of ρ and ω are sufficiently attractive to bind the $T = 0$, $\bar{K}N$ system. On this calculation the signs of the ρ and ω exchange force are fixed by the universality principle of the vector theory of strong interactions. Another interesting point is that the same mechanism predicts that the $T = 1$, KN system is strongly repulsive, in agreement with observation. I want to make another remark which is somewhat more general. It is interesting to conjecture that the isospin dependent force for any low energy scattering is dominated by the exchange of the ρ meson coupled universally to the isospin current. The ρ exchange force is then attractive whenever the isospins are antiparallel and repulsive whenever the isospin are parallel. This rule works remarkably well in five cases examined so far. In the case of s-wave πN scattering essentially the entire isospin dependence is due to the ρ exchange as conjectured by Cini and Fubini and by myself independently and as proved by Hamilton *et al.* In the s-wave $\pi\pi$ scattering case the $T = 0$ seems more attractive than the $T = 2$ state. In low energy $K\bar{K}$ scattering there is some evidence for a strong attractive interaction in the $T = 0$ state as we have just heard. In the $\bar{K}N$ case the currently accepted idea, that the 1405 MeV Y_0^* but not the 1385 MeV Y_1^* is likely to be a $\bar{K}N$ bound state resonance, shows that the $T = 0$ state is more attractive. In the $S = 1$, KN case, the $T = 1$ state is definitely more repulsive than the $T = 0$ state. In general bound states and resonances are more likely for states with lower isospins as you can see from the table of elementary particles and resonances. So there seems to be a correlation between the simplicity of quantum numbers and the possibility of bound or resonant states.

ROSENFELD: (in reply to Good): Good asked how carefully we have looked for doubly charged ($T = 2$) resonances. 1.51 GeV/c $K^- + p \to \Sigma^{\pm}\pi^{\mp}\pi^+\pi^-$ ($E^*_{CM} = 2025$ MeV, 400 events) allowed Alston *et al.*, to explore doubly charged $\Sigma\pi$ combination fairly well up to $\simeq 1600$ MeV. Actually, we saw one bump of perhaps 2 standard deviations at 1560 MeV, but nothing that looked interesting. But I repeat for negative strangeness Berkeley has not looked above 1600 MeV.

TICHO: In reply to Good, we are sensitive to $\Xi\pi$ $T = 3/2$ resonance up to Q values of 170 MeV. Within our statistics we see no evidence for resonances other than the reported Ξ^* of $T = 1/2$.

SANDWEISS: In a paper submitted to this conference, results of a π^+p study at 2 GeV are reported. In about 70 events of $\Sigma^+\pi^+K^0$, $\Sigma^0\,\pi^+K^+$ and $\Sigma^+\pi^0K^+$ no evidence for a $T = 2$ $\Sigma\pi$ resonance was found.

GELL-MANN: If we take the unitary symmetry model with baryon and meson octets, with first order violation giving rise to mass differences, we obtain some rules for supermultiplets. The broken symmetry picture is hard to interpret on any fundamental theoretical basis, but I hope that such a justification may be forthcoming on the basis of analytic continuation of resonant states in isotopic spin and strangeness. Instead of constructing just the inverse Regge function $E\,(J)$, we can consider surfaces $E\,(J,\ I,\ Y$, etc.). Certainly the dynamical equations are as smooth in I and Y as they are in J.

Anyway, we may look at the success of the mass rules:

$$\frac{m_N + m_\Xi}{2} = \frac{3m_\Lambda}{4} + \frac{m_\Sigma}{4}$$

and

$$m_K^2 = \frac{3m_\eta^2}{4} + \frac{m_\pi^2}{4}$$

work fine, while

$$m_M^2 = \frac{3m_\omega^2}{4} + \frac{m_\rho^2}{4}$$

does not work quite so well if $M = K^*$.

Suppose, now we try to incorporate the 3/2-3/2 nucleon resonance into the scheme. The only supermultiplet that does not lead to non-existent resonances in the $K - N$ channels is the *10* representation, which gives 4 states:

$$I = 3/2, \quad S = 0$$
$$I = 1\ , \quad S = -1$$
$$I = 1/2, \quad S = -2$$
$$I = 0\ , \quad S = -3$$

The mass rule is stronger here and yields *equal spacing* of these states. Starting with the resonance at 1238 MeV, we may conjecture that the Y_1^*, at 1385 MeV and the Ξ^* at 1535 MeV might belong to this supermultiplet. Certainly they fulfil the requirement of equal spacing. If $J = 3/2^+$ is really right for these two cases, then our speculation might have some value and we should look for the last particle, called, say, Ω^- with $S = -3$, $I = 0$. At 1685 MeV, it would be metastable and should decay by the weak interactions into $K^- + \Lambda$, $\pi^- + \Xi^0$, or $\pi^0 + \Xi^-$. Perhaps it would explain the old Eisenberg event. A beam of K^- with momentum $\gtrsim 3.5$ GeV/c could yield Ω^- by means of $K^- + p \to K^+ + K^0 + \Omega^-$.

ADAIR: I would like to clarify the statements attributed to me concerning the $\Sigma - K$ parity. The conclusions of Tripp, Ferro-Luzzi, and Watson, and, implicitly, those of Capps, that the $\Sigma - K$ parity is odd, actually refer to a particular model of the K-nucleon interaction, a model which at best can be but a first approximation to the real world, and which does not exhibit certain important features of their data. In particular, this model assumes that the energy dependence of background amplitudes and resonance widths may be neglected, that charge dependent effects may be neglected, and most important, that there are no background amplitudes in the states with the J and parity of the resonance. Elementary considerations show that for the high angular momentum states of interest such approximations are quite inadequate. For example, the magnitude of background amplitudes and widths must vary with energy perhaps by a factor of two, over the resonance region. Their model results in equal resonant cross-sections for the

(Gell-Mann 1962) "If the information I've heard is really right, then our speculation might have some value, and we should look for the last particles, called, say, Omega-minus"

Decuplet skeleton and baryon decuplet

Decuplet skeleton and baryon decuplet

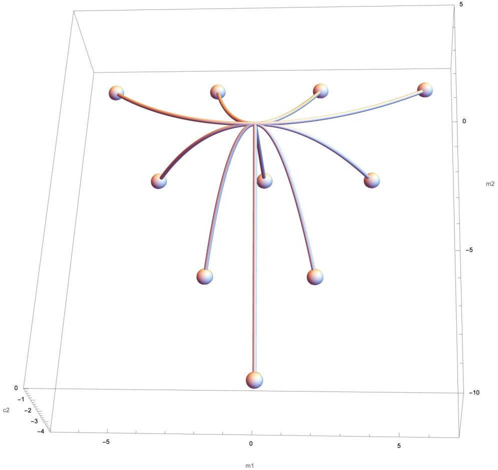
- big algebra of third symmetric power of $SU(3)$

Decuplet skeleton and baryon decuplet

- big algebra of third symmetric power of $\mathrm{SU}(3)$:
  $\mathbb{C}[c_2, c_3, M_1, M_2] / \begin{pmatrix} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18 M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{pmatrix}$

Decuplet skeleton and baryon decuplet

- big algebra of third symmetric power of $\mathrm{SU}(3)$:
$$\mathbb{C}[c_2, c_3, M_1, M_2] / \begin{pmatrix} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{pmatrix}$$

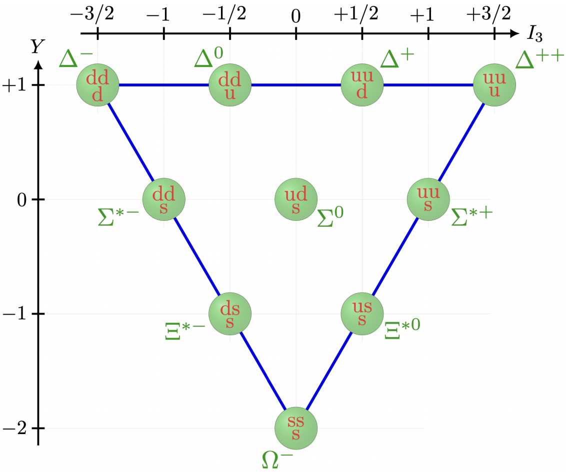


decuplet skeleton over its principal spectrum

Decuplet skeleton and baryon decuplet

- big algebra of third symmetric power of $\mathrm{SU}(3)$:

$$\mathbb{C}[c_2, c_3, M_1, M_2] / \begin{pmatrix} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{pmatrix}$$

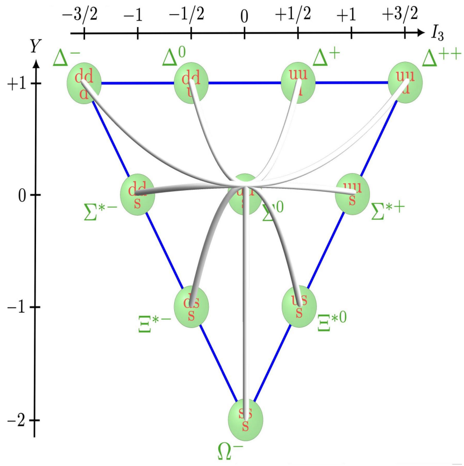


particles in baryon decuplet

Decuplet skeleton and baryon decuplet

- big algebra of third symmetric power of $\mathrm{SU}(3)$:

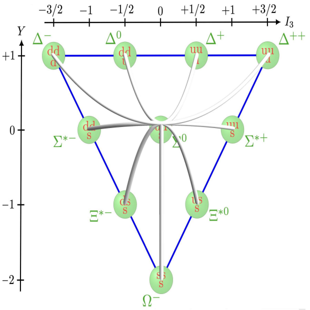
$$\mathbb{C}[c_2, c_3, M_1, M_2] / \begin{pmatrix} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18 M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{pmatrix}$$



skeleton over baryon decuplet

Decuplet skeleton and baryon decuplet

- big algebra of third symmetric power of $\mathrm{SU}(3)$:
  $\mathbb{C}[c_2, c_3, M_1, M_2] / \begin{pmatrix} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{pmatrix}$

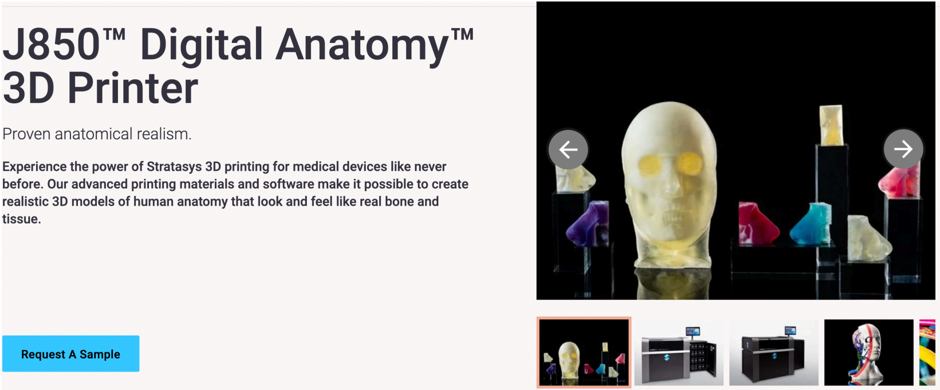


skeleton over baryon decuplet

- $\rightsquigarrow$ relations for $I_3 = \frac{(M_1)_{h_0}}{4}$ and $Y = \frac{(M_2)_{h_0}}{4}$ in baryon decuplet

$$I_3(Y-1)(4I_3^2 - 3Y - 4) = 0$$
$$16I_3^4 - 24I_3^2 Y - 16I_3^2 + 3Y^2 + 6Y = 0$$

Digital anatomy 3D printer



3D printer for anatomy of big algebras

3D printing decuplet crystal